* ## The Divergence of a Vector point function

- Let $\vec{F}(x,y,z)$ be one vector point function, which is defined and differentiable in some region of space.

- Then the divergence of $\vec{F}$ is defined as the scalar product of the vector operator, $\vec{\nabla}$ and $\vec{F}$

and denoted as $div.(\vec{F})$.

(i.e) $\boxed{div(\vec{F}) = \vec{\nabla}\cdot\vec{F}}$

- $\vec{\nabla}\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\vec{F}$

$= \vec{i}\frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$; If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

then $\vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$, which is a scalar quantity.

- If the vector function, $\vec{F}$ represents the velocity of a fluid then the divergence of $\vec{F}$ represents the rate at which the fluid is issued at one point.

- If the fluid is incompressible, $\boxed{\vec{\nabla}\cdot\vec{F} = 0}$ which is known as equation of continuity.

- For any vector point ~~vect~~ function, $\vec{F}$ which satisfies the condition, $\vec{\nabla}\cdot\vec{F} = 0$ is called SOLENOIDAL VECTOR.

THE CURL OF A VECTOR POINT FUNCTION.

- Let $\vec{F}(x,y,z)$ be one vector point function, which is defined and differentiable in some region of space.
- Then the curl of $\vec{F}$ is defined as the vector product of the vector operator, $\vec{\nabla}$ and $\vec{F}$. and denoted as curl $(\vec{F})$.

(i.e) $\boxed{\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F}}$

- If $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$, then

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

- **NOTE :**

(A) → Grad $\phi = \vec{\nabla}\phi$, a vector.

Div$(\vec{F}) = \vec{\nabla} \cdot \vec{F}$, a scalar.

curl$(\vec{F}) = \vec{\nabla} \times \vec{F}$, a vector.

(B) → Curl. grad $\phi = 0$ (always)

(i.e) $\vec{\nabla} \times \vec{\nabla}\phi = 0$

→ div curl $\vec{F} = 0$ (always)

(i.e) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$

(C) → For any $\vec{F}$

if $\vec{\nabla} \times \vec{F} = 0$

(i.e) if curl$(\vec{F}) = 0$

then $\vec{F}$ is called

IRROTATIONAL.

⊙ **Problem ①** Find divergence and curl of $\vec{F} = xyz\,\vec{i} + xyz^2\,\vec{j} + x^2yz\,\vec{k}$

Solution: We know divergence of $\vec{F}$ is defined as

⊛ $\text{div}(\vec{F}) = \vec{\nabla}\cdot\vec{F}$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\cdot\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left(xyz\,\vec{i} + xyz^2\,\vec{j} + x^2yz\,\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(x^2yz)$$

$$= \underline{yz + xz^2 + x^2y}$$

⊛ $$\text{curl}(\vec{F}) = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xyz^2 & x^2yz \end{vmatrix}$$

$$\Rightarrow \text{curl}(\vec{F}) = \vec{i}\left[\partial/\partial y(x^2yz) - \frac{\partial}{\partial z}(xyz^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(x^2yz) - \frac{\partial}{\partial z}(xyz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial y}(xyz)\right]$$

$$\Rightarrow \underline{\text{curl}(\vec{F})} = \underline{\vec{i}(x^2z - 2xyz) - \vec{j}(2xyz - xy) + \vec{k}(yz^2 - xz)}$$

⊙ **Problem: ②** Find 'a' such that

$(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is Solenoidal.

Solution: It is given that the above vector is Solenoidal.

$$\therefore \quad \vec{\nabla}\cdot\vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \partial/\partial z(x - y + 2z) = 0$$

$$\Rightarrow \quad 3 + a + 2 = 0 \Rightarrow \boxed{a = -5}$$

Problem: 3 Show that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational.

For irrotational vectors $\vec{\nabla} \times \vec{F} = 0$.

So let us prove, for given $\vec{F}$, $\vec{\nabla} \times \vec{F} = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\partial/\partial x(zx) - \partial/\partial y(yz)\right]$$

$$= \vec{i}[x - x] - \vec{j}[y - y] + \vec{k}(z - z) = 0.$$

$\therefore$ given $\vec{F}$ is irrotational.

Problem: 4 Find $\text{curl}(\nabla \vec{u})$ when $u = x^2 + y^2 + z^2$.

Solution: Given $u = x^2 + y^2 + z^2 \Rightarrow \vec{\nabla u} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\,\partial/\partial z\right)(x^2 + y^2 + z^2)$

(i.e) $\vec{\nabla u} = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$.

$$\text{curl}(\nabla \vec{u}) = \vec{\nabla} \times \vec{\nabla u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 2x & 2y & 2z \end{vmatrix} = \begin{matrix} \vec{i}\left[\partial/\partial y(2z) - \partial/\partial z(2y)\right] \\ -\vec{j}\left[\partial/\partial x(2z) - \vec{k}\,\partial/\partial z(2x)\right] \\ +\vec{k}\left[\partial/\partial x(2y) - \partial/\partial y(2x)\right] \end{matrix}$$

$\Rightarrow \text{Curl}(\nabla \vec{u}) = 0$

(i.e) $\text{curl}\cdot\text{grad}(u) = 0$, which is true for any scalar point function.

⊙ **Problem: ⑤** Find $div(curl\,\vec{F})$ where $\vec{F} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$

* Let us first find $curl\,\vec{F} = \vec{\nabla}\times\vec{F}$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix} = \vec{i}\left[2z - x\right] - \vec{j}(0) + \vec{k}\left[z - x^2\right]$$

$$\text{(i.e)} \quad \vec{\nabla}\times\vec{F} = \underline{curl\,\vec{F} = (2z - x)\vec{i} + (z - x^2)\vec{k}} \longrightarrow ①$$

* Now we find $div\,(curl\,F)$ using ①.

$$div(curl\,\vec{F}) = \nabla\cdot(\vec{\nabla}\times\vec{F})$$

$$= \vec{\nabla}\cdot\left[(2z - x)\vec{i} + (z - x^2)\vec{k}\right]$$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\partial/\partial z\right)\cdot\left((2z-x)\vec{i} + (z - x^2)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(2z - x) + \frac{\partial}{\partial z}(z - x^2)$$

$$= -1 + 1 = \underline{\underline{0}}$$

$\Rightarrow$ $\underline{div\,(curl\,\vec{F}) = 0}$, which is true for any scalar point function.

⊙ **Problem: ⑥**

Show that $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ is irrotational

Also find scalar potential function φ such that $\vec{F} = \nabla\varphi$.

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ — (1)

* To prove $\vec{F}$ is irrotational, we must show $\vec{\nabla}\times\vec{F} = 0$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = \vec{i}\,[x-x] - \vec{j}\,[y-y] + \vec{k}\,[z-z]$$

$$= 0$$

Since $\vec{\nabla}\times\vec{F} = 0$, $\vec{F}$ is irrotational.

* To find scalar potential function ϕ

it is given " find ϕ such that $\vec{F} = \vec{\nabla}\varphi$

(i.e) $\vec{F} = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$.

$\Rightarrow \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

$\Rightarrow \frac{\partial\varphi}{\partial x} = yz \Rightarrow \varphi = xyz + C_1$ (integrating p. w.r.t. x)

$\Rightarrow \frac{\partial\varphi}{\partial y} = zx \Rightarrow \varphi = xyz + C_2$ (integrating p.w.r.t. y)

$\Rightarrow \frac{\partial\varphi}{\partial z} = xy \Rightarrow \varphi = xyz + C_3$ (integrating p.w.r.t. z)

$\Rightarrow$ The required scalar potential function is

$$\varphi = xyz + C.$$