VECTOR CALCULUS

* Vector calculus deals with two types of vectors
    - Constant Vector ↓ Vectors which are Constant in magnitude and fixed in direction
    - Variable vector. ↓ Vectors which are varying in magnitude (or) direction (or) in both.

* In vector calculus we deal with fields
    - field is one region of Space such that every point, P, in this region is specified with a physical property.

* Fields are of two kinds

FIELDS → Scalar field, Vector field

Scalar field

- Scalar field is the one where the physical Property associated with every point is expressed as scalar quantity.
- This scalar quantity will have different values at different points.
- Its value at any point, P, will depend on the Co-ordinates of P.
- Hence it is called Scalar Point function
- example: Temperature of a swimming pool.

Vector field

- vector field is the one where the physical Property associated with every point is expressed as vector quantity.
- Example: → The velocity at all the points of a moving fluid
- The velocity, at every point, will be represented by a continuous vector function.
- At every point this vector function is specified by a vector with certain magnitude and direction
- Hence it is called Vector point function

* VECTOR DIFFERENTIAL OPERATOR:

The vector operator, ∇ (del) is defined as

$$\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

- It possesses properties analagous to those of ordinary vectors as well as differential operators.
- It is useful in defining
  - Gradient
  - The divergence
  - The curl.

* GRADIENT OF A SCALAR POINT FUNCTION:

- Let $\varphi(x,y,z)$ be a scalar point function defined in some region of space. Then the corresponding <u>Vector point function</u>, $i\frac{\partial\varphi}{\partial x} + j\frac{\partial\varphi}{\partial y} + k\cdot\frac{\partial\varphi}{\partial z}$ is known as the gradient of ϕ and is denoted by $\text{grad}\,\varphi$.

- $$\text{Grad}\,\phi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

  $$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\varphi$$

  $$\text{grad}(\varphi) = \vec{\nabla}\phi$$

- $\vec{\nabla\varphi}$ defines a vector field.
- If φ is constant, then $\nabla\phi = 0$.

  (i.e) $\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial z} = 0$.

$$\nabla\left(\frac{u}{v}\right) = \frac{v\cdot\nabla u - u\nabla v}{v^2}$$

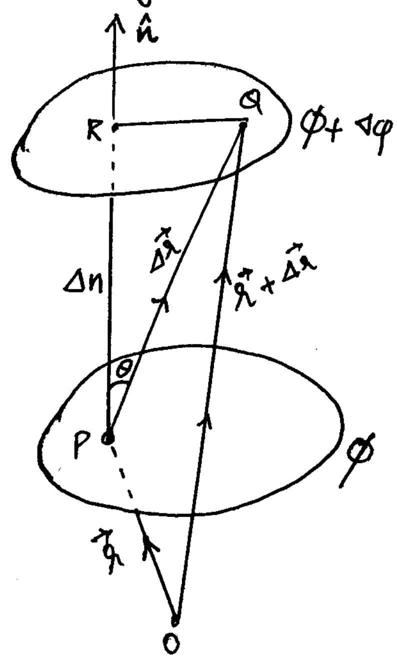
- $\nabla(u+v) = \nabla u + \nabla v$
- $\nabla(uv) = u\cdot\nabla v + v\cdot\nabla u$
- If $v = f(u)$ then $\nabla v = \nabla(f(u))$

  $= f'(u)\cdot\nabla u$

where u and v are two scalar point functions.

* DIRECTIONAL DERIVATIVE OF A SCALAR POINT FUNCTION.

- Let $\phi(x,y,z)$ be a scalar point function defined in some region.
- Let P and Q be two neighbouring points in that region.
- Whose position vectors be $\vec{OP} = \vec{r}$ and $\vec{OQ} = \vec{r} + \Delta\vec{r}$ w.r.t origin O.
- Let (ϕ) and $(\phi + \Delta\phi)$ be the values of the S.P.F at P and Q respectively.
- Let Δr be the length of the vector $\Delta\vec{r}$.



- $\frac{\Delta\phi}{\Delta r}$ is the measure of the rate at which ϕ changes when we move from P to Q.
- The limiting value of this ratio, $\frac{\Delta\phi}{\Delta r}$ as $\Delta r \to 0$ is called directional derivative of ϕ, in the direction PQ.

(i.e) directional derivative of $\phi = \lim_{\Delta r \to 0} \frac{\Delta\phi}{\Delta r} = \frac{d\phi}{dr}$.

* LEVEL SURFACES:

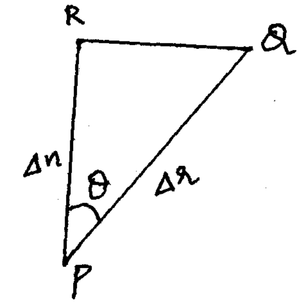
- The surface passing through any arbitrary point P such that, at each point on it, the value of the scalar point function will remain same as in P, is called level surface.

- Consider two such level surfaces passing through P and Q. Let the normal at P, meet the level surface through Q at the point R. Let $PR = \Delta n$ (least distance betn surfaces)

$\frac{d\phi}{dn} \rightarrow$ rate of change of ϕ along normal, PR.

$\frac{d\phi}{dr} \rightarrow$ rate of change of ϕ along $\vec{PQ}$.

- If $Q\hat{P}R = \theta$, $\cos\theta = \dfrac{\Delta n}{\Delta r} \Rightarrow \boxed{\Delta n = \Delta r \cdot \cos\theta}$ → (**)



- We can write $\dfrac{d\varphi}{dr} = \dfrac{d\varphi}{dn} \cdot \dfrac{dn}{dr}$

$$= \frac{d\varphi}{dn} \cdot \cos\theta$$

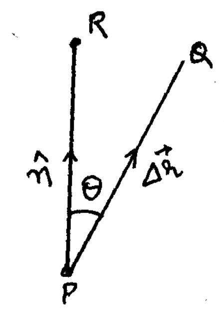
As $\cos\theta \le 1$, we have $\boxed{\dfrac{d\varphi}{dr} \le \dfrac{d\varphi}{dn}} \longrightarrow (*)$

- (*) ⇒ The rate of increase of ϕ along the direction PQ is always less than that along the normal at P.
- At any point the rate of increase of ϕ is greatest only along the normal at the point than any other direction.

* • Let $\hat{n}$ be the unit vector along the normal at P.

- Then $\hat{n} \cdot \Delta\vec{r} = |\hat{n}| \cdot |\Delta\vec{r}| \cos\theta$

$$= 1 \cdot \Delta r \cos\theta$$



i.e.) $\Delta r \cos\theta = \hat{n} \cdot \Delta\vec{r}$ —— (1)

but $\Delta r \cos\theta = \Delta n$ (refer (**)) —— (2)

using (2) in (1) we get $\boxed{\Delta n = \hat{n} \cdot \Delta\vec{r}}$ — (3) $\Rightarrow dn = \hat{n} \cdot d\vec{r}$

- Now we can write $d\varphi = \dfrac{d\varphi}{dn} \cdot dn \Rightarrow \boxed{d\varphi = \dfrac{d\varphi}{dn}\, \hat{n} \cdot d\vec{r}}$ — (4)

- but $d\varphi = \dfrac{\partial\varphi}{\partial x} \cdot dx + \dfrac{\partial\varphi}{\partial y} \cdot dy + \dfrac{\partial\varphi}{\partial z} dz$

$\boxed{d\varphi = \vec{\nabla}\varphi \cdot d\vec{r}}$ — (5) equating (4) & (5)

$$\vec{\nabla}\varphi \cdot d\vec{r} = \frac{d\varphi}{dn}\, \hat{n} \cdot d\vec{r}$$

$$\Rightarrow \boxed{\vec{\nabla}\varphi = \frac{d\varphi}{dn} \cdot \hat{n}} \quad (6)$$

* ⑥ ⇒ $\nabla\varphi$ is a vector in the direction $\hat{n}$ and its magnitude is $\frac{d\varphi}{dn}$ and we know this is the greatest rate of increase of ϕ.

⊙ So we conclude, The gradient of scalar point function, $\varphi(x,y,z)$. is a vector along a normal to the level surface $\varphi(x,y,z)=C$ and its magnitude is the greatest increase of ϕ, namely, $\frac{d\varphi}{dn}$.

(*) ⊙ Since $\frac{d\varphi}{dr} = \frac{d\varphi}{dn}\cos\theta$, The directional derivative of the function (ϕ) in any arbitrary direction is the projection of the gradient of the function in that direction.

⊙ Directional derivative of a function is maximum along the direction of $\vec{\nabla}\varphi$.

(*) Problem:→ In what direction, from the point $(-1, 1, 2)$, is the directional derivative of $\phi = xy^2z^3$ a maximum? What is its magnitude?

Solution: We know that directional derivative is maximum along $\vec{\nabla}\varphi$

∴ Let us find $\vec{\nabla}\varphi$, W.K.T $\vec{\nabla}\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z} \rightarrow (1)$

Given $\phi = xy^2z^3$ ∴ $\frac{\partial\varphi}{\partial x} = y^2z^3$; $\frac{\partial\varphi}{\partial y} = 2xyz^3$; $\frac{\partial\varphi}{\partial z} = 3xy^2z^2$;

using in (1) we get, $\vec{\nabla}\varphi = \vec{i}(y^2z^3) + \vec{j}(2xyz^3) + \vec{k}(3xy^2z^2)$

$$(\vec{\nabla}\varphi)_{(-1,1,2)} = \underrightarrow{8\vec{i} - 16\vec{j} - 12\vec{k}} \quad (2)$$

$\therefore$ for given $\varphi = xy^2z^3$, the directional derivative will be maximum along $\vec{\nabla\varphi} = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

Its magnitude is $|\vec{\nabla\varphi}| = \sqrt{(8)^2+(-16)^2+(-12)^2} = \sqrt{404} = \underline{\underline{2\sqrt{101}}}$ units

Problem: Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Soln: Given $\varphi = x^2yz + 4xz^2 \Rightarrow \dfrac{\partial\varphi}{\partial x} = 2xyz + 4z^2$; $\dfrac{\partial\varphi}{\partial y} = x^2z$;

$\dfrac{\partial\varphi}{\partial z} = x^2y + 8xz$ $\quad\therefore\quad$ $\vec{\nabla\varphi} = \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$

$$(\vec{\nabla\varphi})_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}.$$

Directional derivative of φ is the Projection of $\vec{\nabla\varphi}$ in the given direction

We k.T. Projection of $\vec{\nabla\varphi}$ on $2\vec{i} - \vec{j} - 2\vec{k}$ is

$$\frac{\vec{\nabla\varphi}\cdot(2\vec{i} - \vec{j} - 2\vec{k})}{|2\vec{i} - \vec{j} - 2\vec{k}|}$$

$$= \frac{(8\vec{i} - \vec{j} - 10\vec{k})\cdot(2\vec{i} - \vec{j} - 2k)}{|2\vec{i} - \vec{j} - 2k|}$$

$$= \frac{16 + 1 + 20}{\sqrt{4+1+4}} = \frac{37}{3},$$