UNIT-II

Ordinary Differential Equations (ODE) and applications

Solution of second and higher order linear ODE with constant coefficients – Cauchy's and Legendre's linear equations – Simultaneous first order linear equations with constant coefficients.

Solution of ODE related to electric circuits, bending of beams, motion of a particle in a resisting medium and simple harmonic motion.

—— × ——

A differential equation is an equation which involves differential coefficients or differentials.

The order of a differential equation is the order of the highest differential coefficient present in the equation.

The degree of a differential equation is the degree of the highest differential coefficient (or) derivative occurring in it.

Examples

	Order	Degree
1. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 2y = 0$	2	1
2. $\left(\frac{d^3y}{dx^3}\right)^2 + 4\left(\frac{d^2y}{dx^2}\right)^5 + 8y = 0$	3	2
3. $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{d^3y}{dx^3} = 2e^{2x}$	3	1
4. $\frac{dy}{dx} - 3y = \cos x$ (linear in y)	1	1

Solution of second and higher order linear ODE with constant coefficients

The general form of a linear differential equation of the n^{th} order with constant coefficients is

$$a_0 \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = X, \quad \longrightarrow (1)$$

where $a_0 (\neq 0), a_1, a_2, \ldots, a_n$ are constants and X is a function of x.

If we use the differential operator symbols $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \ldots, D^n = \frac{d^n}{dx^n}$, equation (1) becomes,

$$(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = X \quad \longrightarrow (2)$$

(or)

$$f(D) y = X,$$

where $f(D)$ is a polynomial in D.

When $X=0$, (2) becomes $f(D)y=0$. $\rightarrow$ (3)
The equation (3) is called the homogeneous equation corresponding to (2).

<u>General solution</u> of (2) is $y=u+v$, where $y=u$ is the general solution of (3), that contains n arbitrary constants and $y=v$ is a particular solution of (2), that contains no arbitrary constants.

u is called the <u>complimentary function</u> (C.F.) and v is called the <u>particular integral</u> (P.I.) of the solution of (2).

Complementary Function

To find the C.F. of the solution of equation $(a_0D^n+a_1D^{n-1}+\dots+a_{n-1}D+a_n)y=X$, $\rightarrow$ (*)
we require the general solution of equation $(a_0D^n+a_1D^{n-1}+\dots+a_{n-1}D+a_n)y=0$. $\rightarrow$ (1)

To get the solution of $f(D)y=0$, or, $(a_0D^n+a_1D^{n-1}+\dots+a_n)y=0$, we first write down the <u>auxiliary equation (A.E.)</u>

$$f(m)=0, \text{ or, } a_0m^n+a_1m^{n-1}+\dots+a_n=0, \rightarrow (2)$$

which is obtained by simply replacing D by m in the operator polynomial, and then by equating it to zero.

The auxiliary equation is an n^{th} degree algebraic equation in m.

The solution of equation (1) depends on the nature of roots of the A.E. (2) as explained below.

Case : 1

The roots of the A.E. are real and distinct

Let the roots of the A.E. (2) be $m_1, m_2, \ldots, m_n$. Then the solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

The C.F. of the solution of (*) is given by

$$u = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

Case : 2

The A.E. has real roots, some of which are equal

Let the roots of the A.E. be $m_1, m_1, m_3, m_4, \ldots, m_n$ (the first two roots are equal). Then the solution of equation (1) is

$$y = (c_1 x + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}.$$

If three roots of the A.E. are equal, i.e., if $m_1 = m_2 = m_3$ (say), then the solution is

$$y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + \dots + c_ne^{m_nx}.$$

In general, if r roots of the A.E. are equal, then the solution of ① becomes

$$y = (c_1x^{r-1} + c_2x^{r-2} + \dots + c_{r-1}x + c_r) + c_{r+1}e^{m_{r+1}x} + \dots + c_ne^{m_nx}.$$

Case :3

Two roots of the A.E. are complex.

As complex roots occur in conjugate pairs, let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

Then the solution of ① is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + c_3e^{m_3x} + \dots + c_ne^{m_nx}.$$

Case :4

Two pairs of complex roots of the A.E. are equal.

That is, $m_1 = m_3 = \alpha + i\beta$ and $m_2 = m_4 = \alpha - i\beta$.

Then the solution of ① is,

$$y = e^{\alpha x}\{(c_1x + c_2)\cos\beta x + (c_3x + c_4)\sin\beta x\} + c_5e^{m_5x} + \dots + c_ne^{m_nx}.$$

Particular Integral

The particular integral (P.I.) of the solution of the equation $f(D)y = X$, (1) is the function v, where $y = v$ is a particular solution of (1) containing no arbitrary constants. The P.I. depends on the value of the RHS function X and is defined as $P.I. = \frac{1}{f(D)}X$, where $\frac{1}{f(D)}$ is the inverse operator of $f(D)$.

1. Solve $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

Solution: The given differential equation is

$$(D^3 - 7D - 6)y = 0.$$

The auxiliary equation is

$$m^3 - 7m - 6 = 0.$$

$$\Rightarrow m = -1, -2, 3.$$

$\therefore$ Complementary function, $C.F. = Ae^{-x} + Be^{-2x} + Ce^{3x}$.

Particular integral, $P.I. = 0$.

$\therefore$ The complete solution is

$$y = Ae^{-x} + Be^{-2x} + Ce^{3x},$$

where A, B & C are arbitrary constants.

2. Solve $(D^3-4D^2+4D)y=0$.

Solution

The auxiliary equation is $m^3-4m^2+4m=0$

$$\Rightarrow m=0,2,2.$$

$$C.F.=c_1+(c_2+c_3x)e^{2x}$$

$$P.I.=0$$

$\therefore$ the complete solution is

$$y=c_1+(c_2+c_3x)e^{2x},$$

where c_1, c_2 & c_3 are arbitrary constants.

3. Solve $(D^4+13D^2+36)y=0$.

Solution The auxiliary equation is

$$m^4+13m^2+36=0$$

$$\Rightarrow (m^2+9)(m^2+4)=0$$

$$\Rightarrow m=\pm 3i \ \& \ m=\pm 2i$$

$$C.F.=A\cos 3x+B\sin 3x+C\cos 2x+D\sin 2x$$

$$P.I.=0$$

$\therefore$ The complete solution is

$$y=A\cos 3x+B\sin 3x+C\cos 2x+D\sin 2x,$$

where A, B, C & D are arbitrary constants.

4. Solve $(D^2-2D+4)^2y=0$.

Solution

The auxiliary equation is

$$(m^2-2m+4)^2=0$$

$$\Rightarrow m=\frac{2\pm\sqrt{4-16}}{2}\ (\text{twice})=1\pm\sqrt{3}i,\ 1\pm\sqrt{3}i.$$

$$C.F. = e^{x}\left[(A+Bx)\cos\sqrt{3}x + (C+Dx)\sin\sqrt{3}x\right]$$

$$P.I. = 0$$

$\therefore$ The complete solution is

$$y = e^{x}\left\{(A+Bx)\cos\sqrt{3}x + (C+Dx)\sin\sqrt{3}x\right\},$$

where A, B, C & D are arbitrary constants.

5. Solve $(D^4+m^4)y = 0.$

Solution:

The auxiliary equation is

$$M^4 + m^4 = 0. \quad \longrightarrow (1)$$

(Note that the auxiliary equation is an algebraic equation in M of degree 4.)

The A.E. (1) may take complex roots.

Hence (1) becomes

$$M^4 + m^4 + 2M^2m^2 = 2M^2m^2$$

$$\Rightarrow (M^2+m^2)^2 = 2M^2m^2$$

$$\Rightarrow M^2+m^2 = \pm Mm\sqrt{2}.$$

Case: 1 $\quad M^2+m^2 = \sqrt{2}\,Mm.$ (Taking +ve sign)

$$\Rightarrow M^2+m^2-\sqrt{2}\,Mm = 0$$

$$\Rightarrow M = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

Case: 2 $\quad M^2+m^2 = -\sqrt{2}\,Mm$ (Taking -ve sign)

$$\Rightarrow M^2+m^2+\sqrt{2}\,Mm = 0$$

$$\Rightarrow M = \frac{-m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}.$$

Hence the roots of (1) are

$$M = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}} \;;\; \frac{-m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}.$$

$$C.F. = e^{\frac{m}{\sqrt{2}}x}\left(A\cos\frac{m}{\sqrt{2}}x + B\sin\frac{m}{\sqrt{2}}x\right) + e^{\frac{-m}{\sqrt{2}}x}\left(C\cos\frac{m}{\sqrt{2}}x + D\sin\frac{m}{\sqrt{2}}x\right)$$

P.I. $= 0$.

$\therefore$ The complete solution is,

$$y = e^{\frac{m}{\sqrt{2}}x}\left(A\cos\frac{m}{\sqrt{2}}x + B\sin\frac{m}{\sqrt{2}}x\right) + e^{\frac{-m}{\sqrt{2}}x}\left(C\cos\frac{m}{\sqrt{2}}x + D\sin\frac{m}{\sqrt{2}}x\right),$$

where A, B, C & D are arbitrary constants.

Rules for finding the particular integral (P.I.)

I - Rule

$X = e^{ax}$, where a is a constant.

$$P.I. = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}, \text{ if } f(a) \neq 0.$$

Case of failure :

When $f(a) = 0$, then $(D-a)^r$ is a factor of $f(D)$.

Let $f(D) = (D-a)^r\phi(D)$, where $\phi(a) \neq 0$.

Then
$$P.I. = \frac{1}{(D-a)^r\phi(D)}e^{ax}$$
$$= \frac{1}{\phi(a)}\left[\frac{1}{(D-a)^r}e^{ax}\right] = \frac{1}{\phi(a)}\frac{x^r}{r!}e^{ax}.$$

In particular, $\frac{1}{D-a}e^{ax} = \frac{x}{1!}e^{ax}$ and

$$\frac{1}{(D-a)^2}e^{ax} = \frac{x^2}{2!}e^{ax}$$ and so on.

Another case of failure :

When $f(a) = 0$ but $f'(a) \neq 0$.

Then $\frac{1}{f(D)}e^{ax} = x.\frac{1}{f'(a)}e^{ax}$.

When $f(a) = 0$ and $f'(a) = 0$ but $f''(a) \neq 0$.

Then $\frac{1}{f(D)}e^{ax} = x^2\frac{1}{f''(a)}e^{ax}$.

6. Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Solution :

The A.E. is $m^3 - 6m^2 + 11m - 6 = 0$

$$\Rightarrow m = 1, 2, 3.$$

$$C.F. = Ae^{x} + Be^{2x} + Ce^{3x}.$$

$$P.I. = \frac{1}{D^3 - 6D^2 + 11D - 6}\left(e^{-2x} + e^{-3x}\right)$$

$$= \frac{1}{(D^3 - 6D^2 + 11D - 6)}e^{-2x} + \frac{1}{(D^3 - 6D^2 + 11D - 6)}e^{-3x}$$

$$= -\frac{1}{60}e^{-2x} - \frac{1}{120}e^{-3x} = \frac{-1}{120}\left(2e^{-2x} + e^{-3x}\right).$$

∴ The complete solution is,

$$y = Ae^{x} + Be^{2x} + Ce^{3x} - \frac{1}{120}\left(2e^{-2x} + e^{-3x}\right),$$

where A, B & C are arbitrary constants.

7. Solve $(D^2-a^2)y = e^{ax} - e^{-ax}$.

Solution : The A. E. is $m^2-a^2=0$

$$\Rightarrow m = \pm a$$

$$C.F. = Ae^{ax} + Be^{-ax}.$$

$$P.I. = \frac{1}{(D^2-a^2)}\left(e^{ax} - e^{-ax}\right)$$

$$= \frac{1}{D^2-a^2}\left(e^{ax}\right) - \frac{1}{D^2-a^2}\left(e^{-ax}\right)$$

$$= x.\frac{1}{2D}\left(e^{ax}\right) - x.\frac{1}{2D}\left(e^{-ax}\right)$$

$$= \frac{x}{2}\cdot\frac{e^{ax}}{a} - \frac{x}{2}\left(\frac{e^{-ax}}{-a}\right)$$

$$= \frac{x}{2}\left(\frac{e^{ax}+e^{-ax}}{a}\right)$$

$$= \frac{x}{a}\cosh ax$$

$\therefore$ The complete solution is

$$y = Ae^{ax} + Be^{-ax} + \frac{x}{a}\cosh ax,$$

where A & B are arbitrary constants.

8. Solve $(D+2)(D-1)^2y = e^{-2x} + 2\sinh x$.

Solution : The A. E. is $(m+2)(m-1)^2 = 0$

$$\Rightarrow m = -2, 1, 1.$$

$$C.F. = Ae^{-2x} + (B+Cx)e^{x}.$$

$$P.I. = \frac{1}{(D+2)(D-1)^2}\left(e^{-2x} + 2\sinh x\right)$$

$$= \frac{1}{(D+2)(D-1)^2}\left(e^{-2x} + e^{x} - e^{-x}\right)$$

$$\text{P.I. } 1 = \frac{1}{(D+2)(D-1)^2} e^{-2x}$$

$$= \frac{1}{D+2}\left[\frac{1}{(D-1)^2} e^{-2x}\right]$$

$$= \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{1}{9} \cdot x \cdot \frac{1}{1} e^{-2x}$$

$$= \frac{x}{9} e^{-2x}$$

$$\text{P.I. } 2 = \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{x}\right]$$

$$= \frac{1}{(D-1)^2} \cdot \frac{e^x}{3}$$

$$= \frac{1}{3} x \cdot \frac{1}{2(D-1)} e^x$$

$$= \frac{1}{3} x^2 \cdot \frac{1}{2} e^x$$

$$= \frac{1}{6} x^2 e^x$$

$$\text{P.I. } 3 = \frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$\therefore$ The complete solution is

$$y = Ae^{-2x} + (B + Cx)e^x + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^x + \frac{e^{-x}}{4},$$

where A, B & C are arbitrary constants.

II- Rule

$X = \sin(ax+b)$, or, $\cos(ax+b)$, where a and b are constants. Suppose $f(D)$ is in the form of $\phi(D^2)$.

If $\phi(-a^2) \neq 0$, then $\dfrac{1}{\phi(D^2)} \sin(ax+b) = \dfrac{1}{\phi(-a^2)} \sin(ax+b)$

and $\dfrac{1}{\phi(D^2)} \cos(ax+b) = \dfrac{1}{\phi(-a^2)} \cos(ax+b)$.

Case of failure:

When $\phi(-a^2) = 0$, (D^2+a^2) is a factor of $\phi(D^2)$.

Let $\phi(D^2) = (D^2+a^2)\,\psi(D^2)$, where $\psi(-a^2) \neq 0$.

$$\frac{1}{\phi(D^2)} \sin(ax+b) = \frac{1}{\psi(-a^2)\,(D^2+a^2)} \sin(ax+b)$$

$$= \frac{1}{\psi(-a^2)} \left[\frac{1}{D^2+a^2} \sin(ax+b) \right]$$

$$= \frac{1}{\psi(-a^2)} \left[\frac{x}{2} \times \text{Integral of } \sin(ax+b) \right]$$

Similarly,

$$\frac{1}{\phi(D^2)} \cos(ax+b) = \frac{1}{\psi(-a^2)} \left[\frac{x}{2} \times \text{Integral of } \cos(ax+b) \right].$$

Suppose $f(D)$ is not in the form of $\phi(D^2)$. We have to replace D^2 by $-a^2$, D^3 by $-a^2 D$, D^4 by a^4 etc. After this has been done, $f(D)$ will be in the form $\alpha D+\beta$.

Then $$P.I. = \frac{1}{f(D)} \sin(ax+b)$$

$$= \frac{1}{\alpha D+\beta} \sin(ax+b)$$

$$= \frac{\alpha D-\beta}{-(\alpha^2 a^2+\beta^2)} \sin(ax+b)$$

$$= \frac{-1}{\alpha^2 a^2+\beta^2}\left(\alpha a \cos(ax+b)-\beta \sin(ax+b)\right)$$

Similarly,

$$\frac{1}{f(D)} \cos(ax+b) = \frac{1}{\alpha^2 a^2+\beta^2}\left(\alpha a \sin(ax+b)+\beta \cos(ax+b)\right)$$

Another case of failure :

Suppose $f(D)$ is zero, if we replace D^2 by $-a^2$. Then differentiate $f(D)$ with respect to D and multiply the expression by x.

That is, $\frac{1}{f(D)}\cos(ax+b) = x.\frac{1}{f'(D)}\cos(ax+b)$

when $f'(-a^2) \neq 0$ (replacing D^2 by $-a^2$ in $f'(D)$).

Similarly, $\frac{1}{f(D)}\sin(ax+b) = x.\frac{1}{f'(D)}\sin(ax+b)$

when $f'(-a^2) \neq 0$ (replacing D^2 by $-a^2$ in $f'(D)$).

Suppose $f'(-a^2) = 0$, we again differentiate the reduced denominator in D wrt. D and again multiply the remaining expression by x simultaneously.

9. Solve $(D^3+1)y = \sin(2x+1)$

Solution :

$$C.F. = Ae^{-x} + e^{\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right).$$

$$P.I. = \frac{1}{D^3+1}\sin(2x+1)$$

$$= \frac{1}{1-4D}\sin(2x+1)$$

$$= \frac{1+4D}{(1+4D)(1-4D)}\sin(2x+1) = \frac{1+4D}{65}\sin(2x+1)$$

$$= \frac{1}{65}\left[\sin(2x+1) + 4D(\sin(2x+1))\right]$$

$$= \frac{1}{65}\left[\sin(2x+1) + 8\cos(2x+1)\right]$$

The complete solution is

$$y = Ae^{-x} + e^{-\frac{1}{2}x}\left(B\cos\frac{\sqrt{3}}{2}x + C\sin\frac{\sqrt{3}}{2}x\right),$$

where A, B & C are constants.

10. Find the P.I. of $(D^2+4)y = \cos(2x+3)$.

Solution

$$\text{P.I.} = \frac{1}{D^2+4}\cos(2x+3)$$

Here the denominator vanishes when D^2 is replaced by $-2^2 (=-4)$. Hence we multiply the numerator by x and differentiate the denominator wrt. D.

$$\text{P.I.} = x \cdot \frac{1}{2D}\cos(2x+3)$$

$$= \frac{x}{2}\int \cos(2x+3)\,dx$$

$$= \frac{x}{4}\sin(2x+3).$$

11. Find the P.I. of $(D^2+4D+8)y = \sin(2x+3)$.

Solution :

$$\text{P.I.} = \frac{1}{D^2+4D+8}\sin(2x+3)$$

$$= \frac{1}{4(D+1)}\sin(2x+3)$$

$$= \frac{D-1}{4(D^2-1)}\sin(2x+3)$$

$$= \frac{1}{20}\left\{\sin(2x+3) - 2\cos(2x+3)\right\}.$$

12. Find the P.I. of $(D^2+4)^2 y = \sin(2x+4)$.

Solution :

$$P.I. = \frac{1}{(D^2+4)^2}\sin(2x+4)$$

$$= x.\frac{1}{2(D^2+4).2D}\sin(2x+4).$$

$$= \frac{x}{4}.\frac{1}{D^2+4}.\frac{1}{D}(\sin(2x+4))$$

$$= \frac{x}{4}.\frac{1}{D^2+4}\frac{-\cos(2x+4)}{2}$$

$$= \frac{-x}{8}.\frac{x}{2}\int +\cos(2x+4)\,dx$$

$$= -\frac{x^2}{16}\frac{\sin(2x+4)}{2}$$

$$= -\frac{x^2}{32}\sin(2x+4).$$

III-Rule

$X =$ a polynomial in x.

For simplicity, let $X = x^m$, where m is a positive integer.

$$P.I. = \frac{1}{f(D)}x^m.$$

Rewrite $f(D)$ in terms of a standard binomial expression of the form $(1 \pm \phi(D))$, by taking out the constant term or the lowest degree term from $f(D)$.

For example, if $f(D) = D^2+4$, then $\phi(D) = \frac{D^2}{4}$.

If $f(D) = D^3+D$, then $\phi(D) = \frac{D^3}{D} = D^2$.

Thus $\quad P.I. = \dfrac{1}{aD^{k}\left(1 \pm \phi(D)\right)} x^{m}$

$$= \frac{1}{aD^{k}} \left(1 \pm \phi(D)\right)^{-1} \left(x^{m}\right)$$

Now expand $\left[1 \pm \phi(D)\right]^{-1}$ in a series of ascending powers of D, by using binomial theorem, so that the simplified expansion of $\dfrac{1}{aD^{k}}\left[1 \pm \phi(D)\right]^{-1}$ may contain terms up to D^{m} and then operate by each term on x^{m}.

The ultimate expansion of $\dfrac{1}{f(D)}$ need not be considered beyond the term D^{m}, since $D^{m+1}\left(x^{m}\right) = 0$, $D^{m+2}\left(x^{m}\right) = 0$ and so on.

— X —

$$(1+x)^{-1} = 1 - x + x^{2} - x^{3} + \cdots$$
$$(1-x)^{-1} = 1 + x + x^{2} + x^{3} + \cdots$$
$$(1+x)^{-2} = 1 - 2x + 3x^{2} - 4x^{3} + \cdots$$
$$(1-x)^{-2} = 1 + 2x + 3x^{2} + 4x^{3} + \cdots$$
$$(1+x)^{-3} = 1 - 3x + 6x^{2} - 10x^{3} + 15x^{4} - \cdots$$
$$(1-x)^{-3} = 1 + 3x + 6x^{2} + 10x^{3} + 15x^{4} + \cdots$$

13. Find the P.I. of $(D^3+3D^2+2D)y = x^2$.

Solution:

$$P.I. = \frac{1}{D^3+3D^2+2D}(x^2)$$

$$= \frac{1}{2D} \frac{1}{\left(1+\frac{3D}{2}+\frac{D^2}{2}\right)}(x^2)$$

$$= \frac{1}{2D}\left(1+\frac{3D+D^2}{2}\right)^{-1}(x^2)$$

$$= \frac{1}{2D}\left(1-\frac{3D+D^2}{2}+\frac{9D^2}{4}+O(D^3)\right)(x^2)$$

$$= \frac{1}{2D}\left[x^2-3x+\frac{7}{2}\right]$$

$$= \frac{1}{2}\int\left(x^2-3x+\frac{7}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}-\frac{3x^2}{2}+\frac{7x}{2}\right]$$

$$= \frac{x^3}{6}-\frac{3x^2}{4}+\frac{7x}{4}.$$

14. Find the P.I. of $(D+3)^3 y = 2x^2+x+1$.

Solution:

$$P.I. = \frac{1}{(D+3)^3}(2x^2+x+1)$$

$$= \frac{1}{3^3} \frac{1}{\left(1+\frac{D}{3}\right)^3}(2x^2+x+1)$$

$$= \frac{1}{27}\left(1+\frac{D}{3}\right)^{-3}(2x^2+x+1)$$

$$= \frac{1}{27}\left(1-\frac{3D}{3}+\frac{6D^2}{9}\right)(2x^2+x+1)$$

$$= \frac{1}{27}\left(2x^2+x+1-(4x+1)+\frac{2}{3}\cdot 4\right)$$

$$= \frac{1}{27}\left(2x^2-3x+\frac{8}{3}\right).$$

IV - Rule

$X = e^{ax} V(x)$, where V is a function of x.

$$P.I. = \frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x).$$

$\frac{1}{f(D+a)} V(x)$ is evaluated by using the rule II or III.

15. Find the P.I. of $(D^2-2D+1)y = e^x$.

Solution :

$$P.I. = \frac{1}{D^2-2D+1} \cdot e^x = \frac{1}{(D-1)^2}(e^x \cdot 1)$$

$$= e^x \frac{1}{(D+1-1)^2}(1) = e^x \cdot \frac{1}{D^2}(1)$$

$$= e^x \cdot \frac{x^2}{2}.$$

16. Find the P.I. of $(D^2+1)y = xe^x$.

Solution

$$P.I. = \frac{1}{D^2+1}(xe^x)$$

$$= e^x \frac{1}{(D+1)^2+1}(x)$$

$$= e^x \frac{1}{D^2+2D+2}(x) = \frac{e^x}{2}\left[\left(1+D+\frac{D^2}{2}\right)^{-1}(x)\right]$$

$$= \frac{e^x}{2}\left[1-\left(D+\frac{D^2}{2}\right)\right](x)$$

$$= \frac{e^x}{2}(x-1).$$

17. Obtain the P.I. of $(D^2-2D+1)y = e^x(3x^2-2)$.

Solution :

$$P.I. = \frac{1}{D^2-2D+1}\left\{e^x(3x^2-2)\right\}$$

$$= \frac{1}{(D-1)^2}\left\{e^x(3x^2-2)\right\}$$

$$= e^x \frac{1}{(D+1-1)^2}(3x^2-2)$$

$$= e^x \frac{1}{D^2}(3x^2-2)$$

$$= e^x\left(\frac{x^4}{4}-x^2\right).$$

18. Find the P.I. of $(D^2-4D+3)y = e^x \cos 2x$

Solution :

$$P.I. = \frac{1}{D^2-4D+3}(e^x \cos 2x)$$

$$= e^x \frac{1}{(D+1)^2-4(D+1)+3}(\cos 2x)$$

$$= e^x \frac{1}{D^2-2D}\cos 2x$$

$$= e^x \frac{1}{-4-2D}\cos 2x$$

$$= \frac{-1}{2}e^x \frac{2-D}{4-D^2}\cos 2x$$

$$= \frac{-1}{2}e^x \frac{2-D}{8}\cos 2x$$

$$= \frac{-1}{16}e^x(2\cos 2x + 2\sin 2x)$$

$$= \frac{-1}{8}e^x(\cos 2x + \sin 2x).$$

V-Rule

$X = x . V(x)$, where $V(x)$ is a function of x.

$$P.I. = \frac{1}{f(D)} x . V(x) = x . \frac{1}{f(D)} V(x) + \frac{d}{dD}\left\{\frac{1}{f(D)}\right\} V(x)$$

(or)

$$P.I. = x . \frac{1}{f(D)} V(x) - \frac{f'(D)}{\{f(D)\}^2} V(x).$$

By repeated applications of this rule, we can find the P.I. when $X = x^r V(x)$, where r is a positive integer.

We may adopt the following alternative procedure to find the P.I., where $X = x^r \cos \alpha x$ (or) $X = x^r \sin \alpha x$.

$$P.I. = \frac{1}{f(D)} x^r \cos \alpha x = \frac{1}{f(D)} \left\{\text{Real part of } x^r e^{i\alpha x}\right\}$$

$$= R.P. \text{ of } \left\{\frac{1}{f(D)} x^r e^{i\alpha x}\right\} = R.P. \text{ of } \left\{e^{i\alpha x} . \frac{1}{f(D+i\alpha)} x^r\right\}$$

Similarly $\frac{1}{f(D)} x^r \sin \alpha x = I.P. \text{ of } \left\{e^{i\alpha x} \frac{1}{f(D+i\alpha)} x^r\right\}$.

19. Find the P.I. of $(D^2-1)y = xe^x \sin x$.

Solution

$$P.I. = \frac{1}{D^2-1}(xe^x \sin x) = e^x \cdot \frac{1}{(D+1)^2-1}(x \sin x)$$

$$= e^x \cdot \frac{1}{D^2+2D} x \sin x = e^x\left[I.P. \text{ of } \frac{1}{D^2+2D}(xe^{ix})\right]$$

$$= e^x\left[I.P. \text{ of } e^{ix} \frac{1}{(D+i)^2+2(D+i)}(x)\right]$$

$$= e^x\left[I.P. \text{ of } e^{ix} \cdot \frac{1}{D^2+2(1+i)D+(2i-1)}(x)\right]$$

$$= e^x\left[I.P. \text{ of } \frac{e^{ix}}{2i-1}\left(1+\left\{\frac{2(1+i)D+D^2}{2i-1}\right\}\right)^{-1}(x)\right]$$

$$= e^x\left[I.P. \text{ of } \frac{e^{ix}}{2i-1}\left\{x - \frac{2(1+i)}{2i-1}\right\}\right]$$

$$= e^x\left[I.P. \text{ of } \left\{\frac{-1}{5} e^{ix}(2i+1)\left(x - \frac{(2-6i)}{5}\right)\right\}\right]$$

$$= \frac{-e^x}{5}\left[I.P. \text{ of } e^{ix}(2i+1)\left\{\frac{5x-2(1-3i)}{5}\right\}\right]$$

$$= \frac{-e^x}{5}\left[I.P. \text{ of } (\cos x + i \sin x)\left\{(5x-14)+i(10x+2)\right\}\right]$$

$$= \frac{-e^x}{5}\left[(10x+2)\cos x + (5x-14)\sin x\right].$$

20. Find the P.I. of $(D^2+2D+1)y = x\cos x$.

Solution :

$$P.I. = \frac{1}{(D^2+2D+1)} x\cos x = \frac{1}{(D+1)^2}(x\cos x)$$

$$= x \cdot \frac{1}{(D+1)^2}\cos x - \frac{2(D+1)}{(D+1)^4}(\cos x)$$

$$= x \cdot \frac{1}{-1+2D+1}\cos x - \frac{2}{(D+1)^3}(\cos x)$$

$$= \frac{x}{2}\sin x - \frac{2}{D^3+3D^2+3D+1}(\cos x)$$

$$= \frac{x}{2}\sin x - \frac{2}{-D-3+3D+1}(\cos x)$$

$$= \frac{x}{2}\sin x - \frac{2}{2D-2}(\cos x)$$

$$= \frac{x}{2}\sin x - \frac{(D+1)}{D^2-1}(\cos x)$$

$$= \frac{x}{2}\sin x + \frac{1}{2}(-\sin x + \cos x)$$

$$= \frac{1}{2}(x-1)\sin x + \frac{1}{2}\cos x.$$

VI - Rule

X is a function of x other than e^{ax}, $\sin(ax+b)$, $\cos(ax+b)$ and so on.

$$P.I. = \frac{1}{f(D)} X$$

$$= \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} X$$

(resolving $f(D)$ into linear factors)

$$= \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \dots + \frac{A_n}{D-m_n} \right) X, \rightarrow (1)$$

(using partial fractions)

Consider $\frac{1}{D-m} X = u$ (say)

$\therefore (D-m)u = X \Rightarrow \frac{du}{dx} - mu = X$, which is linear in u of the first order.

Hence $u = \frac{1}{D-m} X = e^{mx} \int e^{-mx} X \, dx.$

Therefore (1) becomes

$$P.I. = A_1 e^{m_1 x} \int e^{-m_1 x} X \, dx + A_2 e^{m_2 x} \int e^{-m_2 x} X \, dx + \dots + A_n e^{m_n x} \int e^{-m_n x} X \, dx.$$

21. Find the P.I. of $(D^2+1)y = \operatorname{cosec} x$.

Solution:

$$P.I. = \frac{1}{D^2+1} \operatorname{cosec} x$$

$$= \frac{1}{(D+i)(D-i)} \operatorname{cosec} x$$

Partial Fractions

$$\frac{1}{(D+i)(D-i)} = \frac{A}{D+i} + \frac{B}{D-i}$$

$\Rightarrow 1 = A(D-i) + B(D+i)$

$\Rightarrow A + B = 0$ and $-Ai + Bi = 1$

$\Rightarrow -Ai - Ai = 1$

$\Rightarrow A = \frac{-1}{2i}$, hence $B = \frac{1}{2i}$.

$$P.I. = \frac{1}{2i}\left(\frac{-1}{D+i} + \frac{1}{D-i}\right) \operatorname{cosec} x$$

$$\therefore y = \frac{-e^{-ix}}{2i}\int \operatorname{cosec} x \, e^{ix} dx + \frac{1}{2i} e^{ix}\int \operatorname{cosec} x \, e^{-ix} dx$$

$$= \frac{-1}{2i}\Big[e^{-ix}\int \operatorname{cosec} x (\cos x + i \sin x) dx$$
$$- e^{ix}\int \operatorname{cosec} x (\cos x - i \sin x) dx\Big]$$

$$= -\frac{1}{2i}\left[e^{-ix}\int(\cot x + i)dx - e^{ix}\int(\cot x - i)dx\right]$$

$$= -\frac{1}{2i}\left[e^{-ix}(\log \sin x + ix) - e^{ix}(\log \sin x - ix)\right]$$

$$= \frac{1}{2i}\left[e^{ix}(\log \sin x - ix) - e^{-ix}(\log \sin x + ix)\right]$$

$$= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i}\right) - x\left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

$$= \log \sin x . \sin x - x \cos x.$$

22. Find the P.I. of $(D^2 + a^2)y = \tan ax$.

Solution:

$$P.I. = \frac{1}{D^2 + a^2}\tan ax = \frac{1}{(D+ia)(D-ia)}\tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\tan ax - \frac{1}{D+ia}\tan ax\right]$$

$$= \frac{1}{2ia}\left[e^{iax}\int\left(\sin ax - i\frac{\sin^2 ax}{\cos ax}\right)dx - e^{-iax}\int\left(\sin ax + i\frac{\sin^2 ax}{\cos ax}\right)dx\right]$$

$$= \frac{-1}{a^2}\log(\sec ax + \tan ax) . \cos ax$$

(Verify !)

Linear Differential Equations with Variable Coefficients

Euler's (or, Cauchy's) Linear Equations

An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_{n-1} x \frac{dy}{dx}$$
$$+ a_n y = Q, \longrightarrow (1)$$

where a_i's are constants and Q is a function of x, is called <u>Cauchy's homogeneous linear equation.</u>

Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$x = e^z$ (or) $z = \log x$. Let $D' = \frac{d}{dz}$.

Hence $Dy = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$

$$\Rightarrow x Dy = \frac{dy}{dz} = D'y \Rightarrow \boxed{x Dy = D'y}$$

$$D^2 y = \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= \frac{-1}{x^2} \frac{dy}{dz} + \frac{1}{x^2} \frac{d^2 y}{dz^2}$$

$$\Rightarrow x^2 D^2 y = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = D'^2 - D'y$$

$$\Rightarrow \boxed{x^2 D^2 y = D'(D'-1)y}$$

Similarly, $x^3D^3y = D'(D'-1)(D'-2)y$ and so on.

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

24. Solve $(x^2D^2+xD+1)y=0$.

Solution

The given diff. eqn. is

$$(x^2D^2+xD+1)y=0. \longrightarrow (1)$$

Put $x=e^z$ so that $z=\log x$.

Let $D'=\frac{d}{dz}$.

Then (1) becomes

$$(D'(D'-1)+D'+1)y=0$$

$$\Rightarrow (D'^2+1)y=0.$$

The A.E. is $m^2+1=0$

$$\Rightarrow m=\pm i$$

$$C.F. = A\cos z + B\sin z$$

$$\Rightarrow C.F. = A\cos(\log x)+B\sin(\log x)$$

$$P.I. = 0$$

Hence the required solution is,

$$y = A\cos(\log x)+B\sin(\log x).$$

25. Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

Solution:

The given equation $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x+\frac{1}{x}\right) \rightarrow (1)$ is a Cauchy's homogeneous linear differential equation.

Put $x = e^z$ so that $z = \log x$.

Let $D' = \frac{d}{dz}$.

Then (1) becomes

$$\left[D'(D'-1)(D'-2) + 2D'(D'-1) + 2\right]y = 10\left(e^z + e^{-z}\right)$$

$$\Rightarrow \left(D'^3 - D'^2 + 2\right)y = 10\left(e^z + e^{-z}\right), \quad \rightarrow (2)$$

which is a linear equation with constant coefficients.

The A. E. of (2) is $m^3 - m^2 + 2 = 0$

$$\Rightarrow m = -1, \ 1 \pm i$$

$$C.F. = Ae^{-z} + e^z\left(B\cos z + C\sin z\right)$$

$$\Rightarrow C.F. = \frac{A}{x} + x\left(B\cos(\log x) + C\sin(\log x)\right).$$

$$P.I. = 10 . \frac{1}{D^3 - D^2 + 2}\left(e^z + e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} \quad \text{(Verify !)}$$

$$= 5x + \frac{2}{x}\log x$$

$\therefore$ The complete solution is

$$y = \frac{A}{x} + x\left(B\cos(\log x) + C\sin(\log x)\right) + 5x + \frac{2}{x}\log x,$$

where A, B & C are arbitrary constants.

Legendre's Linear Differential Equations

An equation of the form

$$a_0(a+bx)^n \frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$\cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X, \rightarrow (1)$$

where a_i's are constants and X is a function of x, is called Legendre's linear differential equation.

Such equation can be reduced to linear differential equations with constant coefficients by the following substitution.

$$a+bx = e^z \text{ so that } z = \log(a+bx).$$

Let $D' = \frac{d}{dz}$

Now,

$$(a+bx)\frac{dy}{dx} = (a+bx)\left\{\frac{dy}{dz}\cdot\frac{dz}{dx}\right\}$$

$$= (a+bx)\left\{\frac{dy}{dz}\cdot\frac{b}{a+bx}\right\}$$

$$= b\frac{dy}{dz}$$

That is, $\boxed{(a+bx)Dy = bD'y,}$ where $D = \frac{d}{dx}$ & $D' = \frac{d}{dz}$.

Similarly $\boxed{(a+bx)^2 D^2y = b^2 D'(D'-1)y}$

$$(a+bx)^3 D^3 y = b^3 D'(D'-1)(D'-2)y$$

$\vdots$

and so on.

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

26. Solve $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\dfrac{dy}{dx}-36y=3x^2+4x+1.$

Solution :

The given equation $\left((3x+2)^2D^2+3(3x+2)-36\right)y=3x^2+4x+1$ is a Legendre's linear differential equation.

Put $3x+2=e^z$ so that $z=\log(3x+2)$

Here $b=3.$ Let $D'=\dfrac{d}{dz}.$

The given equation becomes,

$$\left[3^2D'(D'-1)+3.3D'-36\right]y=3\left(\frac{e^z-2}{3}\right)^2+4\left(\frac{e^z-2}{3}\right)+1$$

$$\Rightarrow (D'^2-4)y=\frac{1}{27}(e^{2z}-1)$$, which is a linear equation with constant coefficients.

Its A.E. is $m^2-4=0.$

$$\Rightarrow m=\pm 2.$$

$$\text{C.F.}=Ae^{2z}+Be^{-2z}$$

$$\Rightarrow \text{C.F.}=A(3x+2)^2+B(3x+2)^{-2}$$

$$\text{P.I.}=\frac{1}{27}\cdot\frac{1}{D'^2-4}(e^{2z}-1)$$

$$=\frac{1}{108}(ze^{2z}+1)$$

$$=\frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

$\therefore$ The complete solution is

$$y = A(3x+2)^2 + B(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right],$$

where A & B are arbitrary constants.

Simultaneous first order linear differential equations with constant coefficients

We now discuss differential equations in which there is one independent variable and two or more than two dependent variables. Such equations are called simultaneous linear equations.

To solve such equations completely, we must have as many simultaneous equations as the number of dependent variables. Here, we shall consider simultaneous linear equations with constant coefficients only.

Let x, y be the two dependent variables and t the independent variable. Consider the simultaneous equations

$$f_1(D)x + f_2(D)y = T_1 \quad \text{and} \quad \rightarrow (1)$$

$$g_1(D)x + g_2(D)y = T_2 \quad \rightarrow (2)$$

where $D = \frac{d}{dt}$ and T_1 and T_2 are functions of t.

Eliminate y from ① and ②, we have

$$\left\{f_1(D)\,g_2(D) - f_2(D)\,g_1(D)\right\} x = g_2(D)T_1 - f_2(T)T_2.$$

That is, $f(D)x = T$, which is a linear equation in x and t and can be solved by the methods already discussed.

Substituting the value of x in either (1) or (2), we get the value of y.

Note that we can also eliminate x to get a linear equation in y and t.

27. Solve
$$\frac{d^2x}{dt^2} + 4x + 5y = t^2$$
$$\frac{d^2y}{dt^2} + 5x + 4y = t + 1.$$

Solution:

Let $D = \dfrac{d}{dt}$.

The given equations become

$$(D^2+4)x + 5y = t^2 \longrightarrow ①$$

and
$$5x + (D^2+4)y = t+1. \rightarrow ②$$

To eliminate y, operating on both sides of ① by (D^2+4) and on both sides of ② by 5 and subtracting, we get

$$\left[(D^2+4)^2 - 25\right]x = (D^2+4)t^2 - 5(t+1)$$
$$\Rightarrow (D^4 + 8D^2 - 9)x = 4t^2 - 5t - 3.$$

Its A.E. is $m^4+8m^2-9=0$

$$\Rightarrow m=\pm 1, \pm 3i.$$

$$C.F. = Ae^{t}+Be^{-t}+C\cos 3t+D\sin 3t.$$

$$P.I. = \frac{1}{D^4+8D^2-9}(4t^2-5t-3)$$

$$= \frac{-1}{9}\left(1-\frac{8D^2}{9}-\frac{D^4}{9}\right)^{-1}(4t^2-5t-3)$$

$$= \frac{-1}{9}\left(4t^2-5t+\frac{37}{9}\right)$$

$$\therefore\ x = Ae^{t}+Be^{-t}+C\cos 3t+D\sin 3t-\frac{1}{9}\left(4t^2-5t+\frac{37}{9}\right).$$

Now, $\frac{dx}{dt} = Ae^{t}-Be^{-t}-3C\sin 3t+3D\cos 3t-\frac{1}{9}(8t-5).$

$$\frac{d^2x}{dt^2} = Ae^{t}+Be^{-t}-9C\cos 3t-9D\sin 3t-\frac{8}{9}$$

Substituting the values of x and $\frac{d^2x}{dt^2}$ in ①, we have $5y = t^2-4x-\frac{d^2x}{dt^2}$

$$\Rightarrow y=\frac{1}{5}\left[-5Ae^{t}-5Be^{-t}+5C\cos 3t+5D\sin 3t + \frac{25}{9}t^2-\frac{20}{9}t+\frac{220}{81}\right]$$

Hence $x = Ae^{t}+Be^{-t}+C\cos 3t+D\sin 3t-\frac{1}{9}\left(4t^2-5t+\frac{37}{9}\right)$

and $y = -Ae^{t}-Be^{-t}+C\cos 3t+D\sin 3t+\frac{1}{9}\left(5t^2-4t+\frac{44}{9}\right).$