Applications of ODE

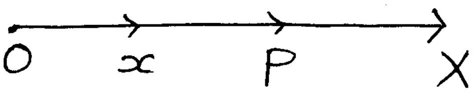
Motion of a particle in a resisting medium

Let a body of mass m start moving from a fixed point O along a straight line OX under the action of a force F. Let P be the position of the body at any instant t, where $OP = x$, then

(i) the velocity, v, of the body $= \frac{dx}{dt}$

(ii) the acceleration, a, of the body $= \frac{dv}{dt}$,

or, $a = \frac{d^2x}{dt^2}$, or, $a = v\frac{dv}{dx}$.



By Newton's second law of motion,

$$F = ma = m\frac{dv}{dt} \text{ or } m\frac{d^2x}{dt^2} \text{ or } mv\frac{dv}{dx},$$

where F is the effective force.

1. A moving body is opposed by a force per unit mass of value cx and resistance per unit mass of value bv^2, where x and v are the displacement and velocity of the particle at that instant. Show that the velocity of the particle, if it starts from rest, is given by $v^2 = \frac{c}{2b^2}(1 - e^{-2bx}) - \frac{cx}{b}$.

Solution :

By Newton's second law of motion, the equation of motion of the body is

$$v\frac{dv}{dx} = -cx - bv^2$$

$$\Rightarrow v\frac{dv}{dx} + bv^2 = -cx \longrightarrow (1)$$

Taking $v^2 = z$, we have $2v\frac{dv}{dx} = \frac{dz}{dx}$.

Hence (1) becomes

$$\frac{1}{2}\frac{dz}{dx} + bz = -cx$$

$$\Rightarrow \frac{dz}{dx} + 2bz = -2cx,$$ which is Leibnitz's linear equation.

$$\Rightarrow (D+2b)z = -2cx, \text{ where } D \equiv \frac{d}{dx}.$$

$$C.F. = Ae^{-2bx}$$

$$P.I. = \frac{1}{D+2b}(-2cx)$$

$$= \frac{(-2c)}{2b}\left(1+\frac{D}{2b}\right)^{-1}(x)$$

$$= -\frac{c}{b}\left(x - \frac{1}{2b}\right) = -\frac{cx}{b} + \frac{c}{2b^2}.$$

The solution is,

$$v^2 = z = Ae^{-2bx} - \frac{cx}{b} + \frac{c}{2b^2}.$$

The given condition is, $v=0$ at $x=0$.

$$0 = A + \frac{c}{2b^2} \Rightarrow A = \frac{-c}{2b^2}.$$

∴ The particular solution is,

$$v^2 = -\frac{c}{2b^2} e^{-2bx} - \frac{cx}{b} + \frac{c}{2b^2}.$$

2. A particle of mass m is projected vertically upward under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}\left[\lambda - \log(1+\lambda)\right]$, where V is the greatest velocity which the above mass will attain when it falls freely and λV is the initial velocity.

Solution:

Let v be the velocity of the particle at time t. The forces acting on the particle are

(i) its weight mg acting vertically downwards

(ii) the resistance mkv of the air acting vertically downwards

Accelerating force on the particle $= -mg - mkv$.

By Newton's second law, the equation of motion of the particle is $mv\frac{dv}{dx} = -mg - mkv$

$$\Rightarrow v\frac{dv}{dx} = -g - kv. \longrightarrow (1)$$

~~When the particle attains the greatest velocity V, its acceleration is zero,~~

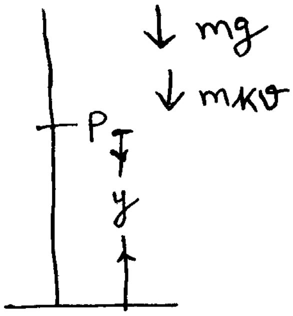
When the particle falls freely (under gravity), equation (1) becomes (changing g to $-g$).
We have
$$v\frac{dv}{dx} = g - kv. \longrightarrow (2)$$

~~That is, $a = g - kv$ [∵ acceleration is zero] ⇒ $k = \frac{g}{v}$~~

When the particle attains the greatest velocity V, its acceleration is zero.

∴ From (2), $0 = g - kV$
$$\Rightarrow k = \frac{g}{V}.$$

Putting this value of k in (1), we have



$$v\frac{dv}{dx} = -g - \frac{g}{V}v = \frac{-g}{V}(V+v)$$
$$\Rightarrow \frac{v\,dv}{V+v} = -\frac{g}{V}dx$$
$$\Rightarrow \int \frac{v}{V+v}dv = \frac{-g}{V}\int dx$$
$$\Rightarrow \int\left(1 - \frac{V}{V+v}\right)dv = \frac{-g}{V}x + A$$
$$\Rightarrow v - V\log(V+v) = \frac{-g}{V}x + A \longrightarrow (3)$$

The given condition is $v = \lambda V$ when $x = 0$.
Hence (3) becomes
$$\lambda V - V\log(V+\lambda V) = \left(\frac{-g}{V}\cdot 0\right) + A = A$$
∴ The particular solution is,
$$v - V\log(V+v) = \frac{-g}{V}x + V\left[\lambda - \log(1+\lambda)V\right]. \longrightarrow (4)$$

Let h be the greatest height attained by the particle, then $x = h$ when $v = 0$.

Hence, from (4), $-V\log V = -\frac{g}{V}h + V[\lambda - \log(1+\lambda)V]$

$$\Rightarrow \frac{g}{V}h = V\lambda - V[\log(1+\lambda)V - \log V]$$

$$\Rightarrow h = \frac{V^2}{g}[\lambda - \log(1+\lambda)].$$

3. A body of mass m, falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity. If it falls through a distance x and possesses a velocity v at that instant, then prove that $\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$, where $mg = ka^2$.

Solution:

The forces acting on the body are
(i) its weight mg acting vertically downwards
(ii) the resistance kv^2 of the air acting vertically upwards.

Accelerating force on the body $= mg - kv^2$
$= ka^2 - kv^2$ $[\because mg = ka^2]$
$= k(a^2 - v^2)$

By Newton's second law, the equation of motion of the body is

$$mv\frac{dv}{dx} = k(a^2 - v^2) \Rightarrow \frac{v}{a^2 - v^2}dv = \frac{k}{m}dx.$$

Integrating, $\int \frac{v}{a^2-v^2}\,dv = \frac{k}{m}\int dx + c$

$$\Rightarrow -\frac{1}{2}\log(a^2-v^2) = \frac{k}{m}x + c \longrightarrow (1)$$

The given initial condition is $v=0$ when $x=0$.

$$\therefore \quad c = -\frac{1}{2}\log a^2.$$

Hence from (1), we have

$$-\frac{1}{2}\log(a^2-v^2) = \frac{k}{m}x - \frac{1}{2}\log a^2$$

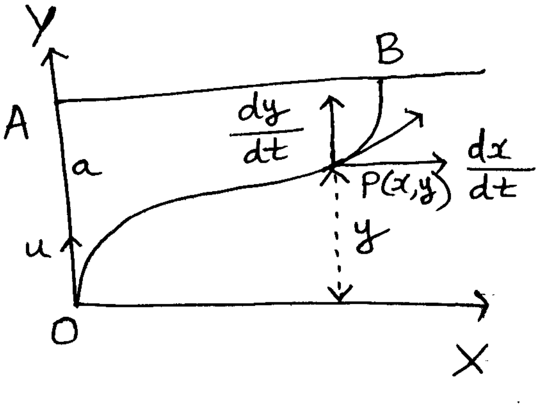
$$\Rightarrow \frac{2kx}{m} = \log a^2 - \log(a^2-v^2)$$

$$\Rightarrow \frac{2kx}{m} = \log\left(\frac{a^2}{a^2-v^2}\right).$$

4. A boat is rowed with a velocity u across a stream of width a. If the velocity of the current is directly proportional to the product of the distances from the two banks, find the equation of the path of the boat and the distance down stream to the point, where it lands.

Solution:

Let the point from where the boat starts be taken as the origin O and the axes as shown in the figure.



At any time t, let the boat be at $P(x,y)$, then its distances from the two banks are y and $(a-y)$.

$$\frac{dx}{dt} = \text{velocity of the current} = ky(a-y)$$

$$\frac{dy}{dt} = \text{velocity of the boat} = u.$$

$$\therefore \quad \frac{dy}{dx} = \frac{u}{ky(a-y)} \longrightarrow (1)$$

This is the direction of the resultant velocity of the boat at P and hence the direction of the tangent to the path of the boat at P.

From (1), $y(a-y)\,dy = \frac{u}{k}\,dx$

Integrating, $\frac{ay^2}{2} - \frac{y^3}{3} = \frac{u}{k}x + c.$

At $x=0$ and $y=0$, $c=0.$

$\therefore$ The equation of the path of the boat is

$$x = \frac{k}{6u}y^2(3a-2y). \longrightarrow (2)$$

The distance down-stream to the point, where the boat lands, is obtained by putting $y=a$ in (2)

Thus $x = \frac{ka^3}{6u}.$

5. A particle of mass moves in a straight line under no forces except a resistance mkv^3, where v is the velocity and k is constant. If the initial velocity is u and x is the distance covered in time t, prove that $kx = \frac{1}{v} - \frac{1}{u}$, $t = \frac{x}{u} + \frac{kx^2}{2}.$

Solution:

Acceleration $a = -\kappa v^3$. But $a = v\dfrac{dv}{dx}$.

$\therefore\ v\dfrac{dv}{dx} = -\kappa v^3 \Rightarrow \dfrac{dv}{v^2} = -\kappa\, dx$

Integrating, $\displaystyle\int_u^v \frac{dv}{v^2} = -\kappa\int_0^x dx,$

where u is the initial velocity.

$$\Rightarrow \frac{1}{v} - \frac{1}{u} = +\kappa x$$

$$\Rightarrow v = \frac{dx}{dt} = \frac{1}{\left(\kappa x + \frac{1}{u}\right)}$$

Hence $\displaystyle\int_0^t dt = \int_0^x \left(\kappa x + \frac{1}{u}\right) dx$

$$\Rightarrow t = \frac{x}{u} + \frac{\kappa x^2}{2}.$$

6. A particle is released from rest at a height α above a horizontal plane. The resistance to the motion of the particle is κv^2 per unit mass, where v is the velocity and κ is constant. Show that the particle strikes the plane with velocity V where $\kappa V^2 = g\left(1 - e^{-2\kappa\alpha}\right)$. If the particle rebounds from the plane without loss of energy and reaches a maximum height β in the subsequent motion, show that $e^{-2\kappa\alpha} + e^{2\kappa\beta} = 2$.

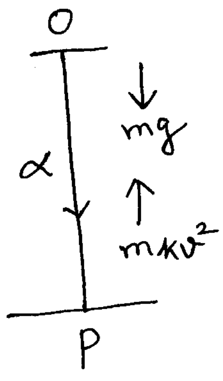
Solution :

Considering the motion downwards, the equation of motion of the particle is

$$ma = mg - mkv^2. \longrightarrow (1)$$

Since $a = v\dfrac{dv}{dx}$, (1) becomes $v\dfrac{dv}{dx} = g - kv^2$.

$$\Rightarrow dx = \frac{v\,dv}{g - kv^2}$$



Integrating, $\displaystyle\int_0^{\alpha} dx = \int_0^{V} \frac{v\,dv}{g - kv^2}$

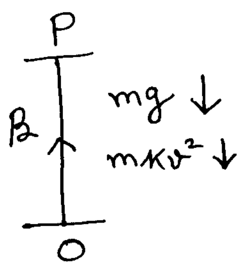
$$\Rightarrow \alpha = \frac{-1}{2k}\left(\log\left(g - kv^2\right)\right)_0^V$$

$$= \frac{-1}{2k}\log\left(\frac{g - kV^2}{g}\right)$$

$$\Rightarrow e^{-2k\alpha} = \frac{g - kV^2}{g}$$

$$\Rightarrow kV^2 = g\left(1 - e^{-2k\alpha}\right) \longrightarrow (2)$$

Now considering the rebound (the upward motion) we choose a fresh origin of coordinates at the plane. Then the equation of motion is



$$ma = -mg - mkv^2$$

$$\Rightarrow v\frac{dv}{dx} = -g - kv^2$$

$$\Rightarrow -dx = \frac{v\,dv}{g + kv^2}$$

Integrating, $\displaystyle -\int_0^{\beta} dx = \int_V^{0} \frac{v\,dv}{g + kv^2}$

$$\Rightarrow \beta = \frac{-1}{2k}\left(\log(g+kv^2)\right)_V^0$$

$$= \frac{-1}{2k}\log\left(\frac{g}{g+kV^2}\right)$$

$$\Rightarrow e^{2k\beta} = \frac{g+kV^2}{g}$$

$$\Rightarrow kV^2 = g\left(e^{2k\beta}-1\right). \longrightarrow (3)$$

From (2) and (3), we have

$$g\left(1-e^{-2k\alpha}\right) = g\left(e^{2k\beta}-1\right)$$

$$\Rightarrow e^{-2k\alpha}+e^{2k\beta} = 2.$$

7. A particle of mass m falls from rest exerting a resistance $\frac{mgv^4}{c^4}$ where v is the speed and c is a constant. Show that it acquires a speed $\frac{c}{2}$ after falling a distance $\frac{c^2}{4g} - \log\left(\frac{5}{3}\right)$ in a time $\frac{c\left[2\tan^{-1}(1/2)+\log 3\right]}{4g}$.

Solution :

The equation of motion of a falling particle is $ma = mg - \frac{mgv^4}{c^4}$.

$$\Rightarrow v\frac{dv}{dx} = g\left[1-\frac{v^4}{c^4}\right] = g\left(\frac{c^4-v^4}{c^4}\right).$$

$$\Rightarrow g\,dx = c^4 \cdot \frac{v\,dv}{c^4 - v^4}$$

$$= c^4 \left(\frac{A}{c-v} + \frac{B}{c+v} + \frac{Cv+D}{c^2+v^2} \right) dv$$

Using partial fraction method, we get

$$A = \frac{1}{4c^2},\ B = -\frac{1}{4c^2},\ C = \frac{1}{2c^2} \text{ and } D = 0.$$

$$\Rightarrow g\,dx = \frac{c^2}{4}\left[\frac{dv}{c-v} - \frac{dv}{c+v} + \frac{2v\,dv}{c^2+v^2} \right]$$

Integrating we get

$$gx = \frac{c^2}{4}\left(-\log(c-v) - \log(c+v) + \log(c^2+v^2) \right)$$

$$= \frac{c^2}{4} \log\left(\frac{c^2+v^2}{c^2-v^2} \right)$$

When $v = \frac{c}{2}$, $gx = \frac{c^2}{4} \log\left(\frac{5}{3}\right)$.

8. A particle of mass m is free to move along a line and is attracted to a fixed point on the line, the attraction being k times the distance of the particle from the fixed point. At time $t=0$, it is distrubed from its equilibrium position by the sudden application of a force $F \sin nt$. Discuss the subsequent motion of the particle, if n is close to the number ω given by $\sqrt{\frac{k}{m}}$.

Solution: The equation of motion is

$$m\frac{d^2x}{dt^2} + kx = F \sin nt$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F}{m} \sin nt$$

~~Solution~~ $\Rightarrow \dfrac{d^2x}{dt^2} + \omega^2 x = \dfrac{F}{m} \sin nt. \longrightarrow (1)$

$$C.F. = A \cos \omega t + B \sin \omega t.$$

$$P.I. = \frac{1}{(D^2+\omega^2)} \frac{F}{m} \sin nt, \text{ where } D = \frac{d}{dt}$$

$$= \frac{F}{m} \cdot \frac{1}{\omega^2 - n^2} \sin nt$$

Hence the general solution is given by

$$x = A \cos \omega t + B \sin \omega t + \frac{F \sin nt}{m(\omega^2 - n^2)} \rightarrow (1)$$

$$\Rightarrow \frac{dx}{dt} = -A\omega \sin \omega t + B\omega \cos \omega t + \frac{Fn \cos nt}{m(\omega^2 - n^2)}$$

The given condition is $x = 0$ and $\dfrac{dx}{dt} = 0$ when $t = 0$.

Hence $A = 0$ and

$$0 = B\omega + \frac{Fn}{m(\omega^2 - n^2)} \Rightarrow B = \frac{-Fn}{m\omega(\omega^2 - n^2)}$$

Using A and B in (1), we have

$$x = \frac{F}{m(\omega^2 - n^2)} \left\{ \sin nt - \frac{n}{\omega} \sin \omega t \right\}.$$

Since by hypothesis, ω is close to n, we can set $\dfrac{n}{\omega} = 1$, as a first approximation.

Then

$$x = \frac{F(\sin nt - \sin \omega t)}{m(\omega^2 - n^2)} \text{ (nearly)}.$$

$$\Rightarrow x = \frac{2F}{m(\omega^2 - n^2)} \cos\left(\frac{n+\omega}{2} t\right) \sin\left(\frac{n-\omega}{2} t\right) \text{ (nearly)}.$$

If we now denote the small quantity $n-\omega$ by 2ϵ and $n+\omega$ is approximately equal to $2n$, we get

$$x = \frac{2F}{m \times 2n \times (-2\epsilon)} \cos(nt) \sin(\epsilon t) \quad \text{(nearly)}$$

$$\Rightarrow x = -\frac{F}{2mn\epsilon} \sin(\epsilon t) \cos(nt) \quad \text{(nearly)}.$$

Since ϵ is a small quantity, the period $\frac{2\pi}{\epsilon}$ of the term $\sin(\epsilon t)$ is large. Hence the form of the last expression shows that x can be regarded as essentially a periodic function $\cos nt$, with slowly varying amplitude $-\frac{F \sin(\epsilon t)}{2mn\epsilon}$.

9. A particle falls in a vertical line under gravity (supposed constant) and the force of air resistance to its motion is proportional to its velocity. Show that its velocity cannot exceed a particular limit.

Solution :

Let V be the velocity when the particle has fallen a distance S in time t from rest. If the resistance is kV, then the equation of motion is

$$\frac{dV}{dt} = g - kV \Rightarrow \frac{dV}{g-kV} = dt.$$

Integrating $-\frac{1}{k}\log(g-kV) = t + C. \rightarrow (1)$

The given initial condition is $V=0$ at $t=0$.

Hence $C = -\frac{1}{k}\log g$.

$\therefore$ (1) becomes $-\frac{1}{k}\log(g-kV) = t - \frac{1}{k}\log g$

$$\Rightarrow t = -\frac{1}{k}\log\left(\frac{g-kV}{g}\right)$$

$$\Rightarrow e^{-kt} = \frac{g-kV}{g}$$

$$\Rightarrow kV = -ge^{-kt} + g$$

$$\Rightarrow V = \frac{g}{k}\left(1-e^{-kt}\right)$$

Since t is positive, e^{-kt} belongs to $(0,1)$.

That is, $\left(1-e^{-kt}\right) < 1$.

$\therefore$ Limiting velocity or Maximum velocity is

$$V = \frac{g}{k}.$$

10. A particle falls under gravity in a medium whose resistance is mk times the velocity. Find (i) the velocity at any time (ii) the distance travelled (iii) also get the relation between the velocity and distance.

Solution :

Take the starting point O as the origin and the downward vertical fall as the Y-axis.

Then the forces acting on the particle are

(i) its weight vertically downwards

(ii) the air resistance mkv upwards.

$\therefore$ The equation of motion is,

$$m\frac{d^2y}{dt^2} = mg - mkv$$

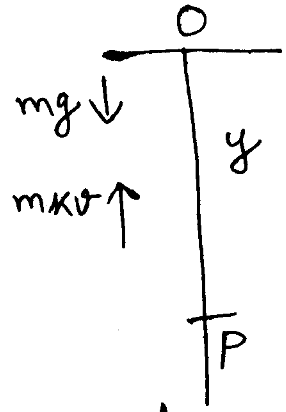
$$\Rightarrow \frac{d^2y}{dt^2} = g - kv$$

$$\Rightarrow \frac{d^2y}{dt^2} = k(V - v), \text{ where } V = \frac{g}{k} \text{ is the} \quad \rightarrow (1)$$

terminal velocity of the medium.

To get the relation between velocity and time, (1) can be rewritten as

$$\frac{dv}{dt} = k(V - v)$$



$$\Rightarrow \frac{dv}{V - v} = k\,dt$$

Integrating, we have

$$-\log(V - v) = kt + A$$

$$\Rightarrow V - v = A_1 e^{-kt}, \text{ where } A_1 = e^{A}.$$

When $t = 0$, $y = 0$. $\therefore V = A_1$

$$\therefore v = V\left[1 - e^{-kt}\right] \longrightarrow (2)$$

(ii) To obtain the distance y in terms of t.

From (2), $v = V(1 - e^{-kt})$

$$\Rightarrow \frac{dy}{dt} = V(1 - e^{-kt})$$

$$\Rightarrow dy = V(1 - e^{-kt})\,dt$$

Integrating, we have

$$y = V\left(t + \frac{e^{-kt}}{k}\right) + B$$

When $t=0$, $y=0$.

$$\therefore\ B = -\frac{V}{k}$$

Hence $$y = \frac{V}{k}\left(e^{-kt} - 1\right) + Vt. \longrightarrow (3)$$

(iii) To obtain v in terms of y.

From (1), $$\frac{d^2y}{dt^2} = k(V-v)$$

But $\frac{d^2y}{dt^2} = v\frac{dv}{dy}$ where $v = \frac{dy}{dt}$

$$\therefore\ v.\frac{dv}{dy} = k(V-v)$$

$$\Rightarrow \int \frac{v\,dv}{V-v} = k\int dy + C$$

$$\Rightarrow -v - V\log(V-v) = ky + C$$

When $y=0$, $v=0$.

$$\therefore\ C = -V\log V$$

$$\therefore\ ky = -v - V\log(V-v) + V\log V$$

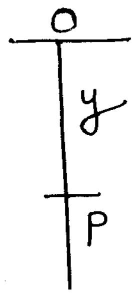
$$\Rightarrow ky = -v + V\log\left(\frac{V}{V-v}\right)$$

$$\therefore\ y = -\frac{v}{k} + \frac{V}{k}\log\left(\frac{V}{V-v}\right).$$

11. A particle of mass m falling under gravity is experiencing a resistance equal to $\frac{mgv^2}{k^2}$, where v is the velocity of the particle. Find its velocity and distance travelled at time t given that initially it starts from rest.

Solution :

Let y be the distance travelled at time t from the initial point O which is taken as the origin of measurement and the downward vertical be taken as the y-axis.



The forces acting on the particle are

(i) its weight mg downwards

(ii) the resistance $\frac{mgv^2}{k^2}$ upwards.

Hence according to Newton's law of motion,

mass x acceleration = resultant force

$$\Rightarrow m.\frac{dv}{dt} = mg - \frac{mgv^2}{k^2}.$$

$$\Rightarrow \frac{dv}{k^2 - v^2} = \frac{g}{k^2}dt$$

Integrating, we have

$$\frac{1}{k}\tanh^{-1}\left(\frac{v}{k}\right) = \frac{g}{k^2}t + A$$

The particle starts from rest.

That is, $v=0$ at $t=0$.

$$\therefore \quad A = 0$$

$$\therefore \frac{1}{k}\tanh^{-1}\frac{v}{k} = \frac{gt}{k^2}$$

$$\Rightarrow \tanh^{-1}\frac{v}{k} = \frac{gt}{k}$$

Hence $v = k\tanh\left(\frac{gt}{k}\right)$.

$$\Rightarrow \frac{dy}{dt} = k\tanh\left(\frac{gt}{k}\right)$$

$$\Rightarrow dy = k\tanh\left(\frac{gt}{k}\right)dt$$

Integrating, we have

$$y = k\,\frac{\log\cosh\left(\frac{gt}{k}\right)}{g/k} + B$$

$$\Rightarrow y = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right) + B$$

When $t=0$, $y=0$.

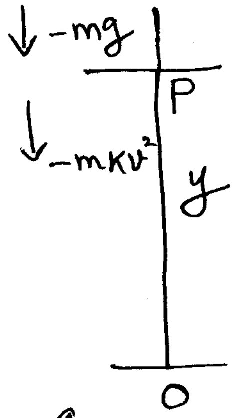
$$\therefore B = 0.$$

Hence $y = \frac{k^2}{g}\log\cosh\left(\frac{gt}{k}\right)$.

12. A particle is projected vertically upwards in a medium where the air resistance varies as the square of the velocity and the initial velocity is v_0. Show that the displacement at any time t is given by $2ky = \log\left(\frac{V^2+v_0^2}{V^2+v^2}\right)$, where $V^2 = \frac{g}{k}$ and also prove that $kt = \frac{1}{V}\left[\tan^{-1}\left(\frac{v_0}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right]$.

Solution:

The particle is moving upwards. The forces acting on the particle are its weight mg vertically downwards, and the air resistance mkv^2 acting downwards. Let the vertical through the point of projection O upwards be taken as the y-axis and let at time t, y be the distance travelled by the particle.



The equation of motion of the particle is

$$m\frac{d^2y}{dt^2} = -mg - mkv^2$$

$$\Rightarrow \frac{d^2y}{dt^2} = -g - kv^2$$

$$= -k\left[V^2+v^2\right], \text{ where } V^2 = \frac{g}{k}$$

$$\therefore v\frac{dv}{dy} = -k\left(V^2+v^2\right) \quad \left[\because \frac{d^2y}{dt^2} = v\frac{dv}{dy}\right]$$

$$\Rightarrow \frac{v\,dv}{V^2+v^2} = -k\,dy$$

Integrating, we have

$$\frac{1}{2}\log\left(V^2+v^2\right) = -ky + A$$

When $t=0$, $y=0$ and $v=v_0$.

$$\therefore \frac{1}{2}\log\left(V^2+v_0^2\right) = A.$$

Hence $$-ky = \frac{1}{2}\log\left(V^2+v^2\right) - \frac{1}{2}\log\left(V^2+v_0^2\right)$$

$$\Rightarrow 2ky = \log\left(\frac{V^2+v_0^2}{V^2+v^2}\right).$$

We have $m\frac{dv}{dt} = -mg - mkv^2$

$$\Rightarrow \frac{dv}{dt} = -k(V^2+v^2), \text{ where } V^2 = \frac{g}{k}$$

$$\Rightarrow \frac{dv}{V^2+v^2} = -k\,dt$$

Integrating, we have

$$\frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) = -kt + B.$$

When $t=0$, $v=V_0$.

$$\therefore \quad B = \frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right)$$

Hence $$kt = \frac{1}{V}\left[\tan^{-1}\left(\frac{V_0}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right].$$

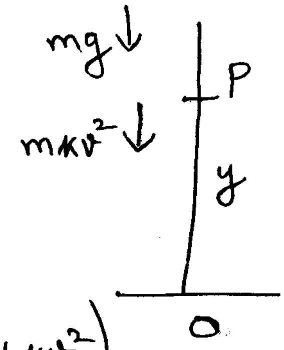
13. A particle is projected vertically upwards with a velocity u and the resistance of air develops a retardation kv^2 where v is the velocity at time t. Find the (i) greatest height reached by the particle and (ii) the velocity with which the particle returns to the point of projection.

Solution :

Taking the upward verticle through the point of projection as the y-axis and the point of projection as the origin, the forces acting on the particle for the vertical upward motion are (i) mg downwards (ii) kmv^2 downwards.

(i) The equation of motion for the upward travel is,

$$m\frac{dv}{dt} = -mg - mkv^2$$



$$\therefore \frac{dv}{dt} = -(g + kv^2)$$

$$\Rightarrow v\frac{dv}{dy} = \frac{dv}{dy}\cdot\frac{dy}{dt} = \frac{dv}{dt} = -(g+kv^2)$$

$$\Rightarrow v\frac{dv}{dy} = -k\left[V^2+v^2\right] \text{ where } V^2 = \frac{g}{k}$$

and V is the terminal velocity of the medium.

$$\Rightarrow \frac{v\,dv}{V^2+v^2} = -k\,dy$$

$$\Rightarrow \frac{1}{2}\log(V^2+v^2) = -ky + A$$

When $y=0$, $v=u$.

$$\therefore A = \frac{1}{2}\log(V^2+u^2)$$

$$\text{Hence } ky = \frac{1}{2}\log\left(\frac{V^2+u^2}{V^2+v^2}\right)$$

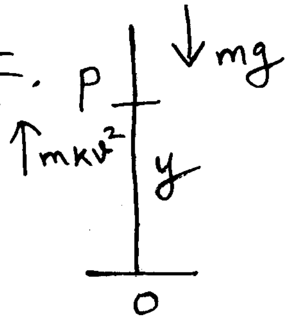
$$\Rightarrow e^{2ky} = \frac{V^2+u^2}{V^2+v^2}$$

Let h be the greatest height reached by the particle so that $v=0$ at that instant.

$$\therefore \frac{V^2+u^2}{V^2} = e^{2kh} \Rightarrow h = \frac{1}{2k}\log\left(\frac{V^2+u^2}{V^2+v^2}\right). \rightarrow (*)$$

(ii) For the downward motion (measuring y still upwards) the equation of motion is

$$m\frac{d^2y}{dt^2} = -mg + k.mv^2$$



$$\Rightarrow \frac{d^2y}{dt^2} = -k(V^2-v^2), \text{ where } V^2 = \frac{g}{k}.$$

$$\Rightarrow v.\frac{dv}{dy} = -k(V^2-v^2)$$

$$\Rightarrow v\,dv = -k(V^2-v^2)\,dy$$

Integrating, $\Rightarrow \dfrac{-v\,dv}{V^2-v^2} = +k\,dy$

$$\frac{1}{2}\log(V^2-v^2) = ky + B.$$

When $y=h$, $v=0$.

$$\frac{1}{2}\log V^2 = kh + B$$

Hence $\dfrac{1}{2}\log(V^2-v^2) = ky + \dfrac{1}{2}\log V^2 - kh$

$$\Rightarrow -2k(y-h) = \log\left(\frac{V^2-v^2}{V^2}\right)$$

$$\Rightarrow 2k(y-h) = \log\left(\frac{V^2}{V^2-v^2}\right)$$

Let U be the velocity with which the particle reaches the ground or point of projection $y=0$.

That is, $v=U$ when $y=0$.

$$2kh = \log\left(\frac{V^2}{V^2-U^2}\right) \Rightarrow e^{2kh} = \frac{V^2}{V^2-U^2}$$

From equation (*), we have $e^{2kh} = \dfrac{V^2+u^2}{V^2}$.

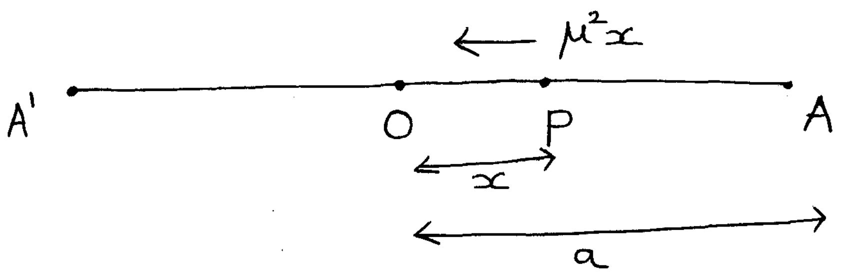
$$\frac{V^2}{V^2-U^2} = \frac{V^2+u^2}{V^2}$$

$$\Rightarrow \frac{1}{U^2} = \frac{u^2+V^2}{u^2V^2}$$

$$\Rightarrow \frac{1}{U^2} = \frac{1}{u^2} + \frac{1}{V^2}.$$

Simple Harmonic Motion (SHM)

A particle is said to execute simple harmonic motion if it moves in a straight line such that its acceleration is always directed towards a fixed point in the line and is proportional to the distance of the particle from the fixed point.



Let O be the fixed point in the line A'A. Let P be the position of the particle at any time t, where $OP = x$.

Since the acceleration is always directed towards O (that is, the acceleration is in the direction opposite to that in which x increases), the equation of motion of the particle is $\frac{d^2x}{dt^2} = -\mu^2 x$.

$$\Rightarrow (D^2 + \mu^2)x = 0, \text{ where } D \equiv \frac{d}{dt}.$$

$$\therefore\ x = A_1 \cos \mu t + A_2 \sin \mu t. \rightarrow (1)$$

Velocity at particle at $P = \frac{dx}{dt} = -A_1 \mu \sin \mu t + A_2 \mu \cos \mu t \rightarrow (2)$

If the particle starts from rest at A, then $A_1 = a$ ($\because$ at $t=0$, $x=a$)

and $A_2 = 0$ ($\because$ at $t=0$, $\frac{dx}{dt} = 0$).

$$\therefore \; x = a\cos\mu t, \qquad \rightarrow (3)$$

and $$\frac{dx}{dt} = -a\mu\sin\mu t. \qquad \rightarrow (4)$$

$$= -a\mu\sqrt{1-\cos^2\mu t} = -a\mu\sqrt{1-\frac{x^2}{a^2}}$$

$$= -\mu\sqrt{a^2-x^2}. \qquad \rightarrow (5)$$

Equation (3) gives the displacement of the particle from the fixed point O at any time t.

Equation (5) gives the velocity of the particle at any time t, when its displacement from the fixed point O is x.

Equation (5) also shows that the velocity is directed towards O and decreases as x increases.

Now, equations (3) and (4) remain unaltered, when t is replaced by $t+\frac{2\pi}{\mu}$, that is, when t is increased by $\frac{2\pi}{\mu}$ showing thereby that the particle occupies the same position and has the same velocity after a time $\frac{2\pi}{\mu}$. the quantity $\frac{2\pi}{\mu}$, usually denoted by T, is called the particle time, that is, the time of complete oscillation.

Nature of motion

At A, $x = a$ and $v = 0$. Since acceleration is directed towards O, the particle moves towards O. The acceleration gradually decreases and vanishes at O, when the particle has acquired maximum velocity. Thus the particle moves further towards A' under retardation and comes to rest at A', where $OA' = OA$. It moves back towards O under acceleration and acquires maximum velocity at O. Thus the particle moves further towards A under retardation and comes to rest at A.

It retraces its path and goes on oscilating between A and A'. The point O is called the centre of motion or the mean position. The maximum distance a which the particle covers on either side of the mean position is called the amplitude of the motion.

The number of complete oscillations per second is called the frequency of motion. If n is the frequency, then

$$n = \frac{1}{T} = \frac{\mu}{2\pi}.$$

1. A particle is executing simple harmonic motion with amplitude 20 cm and time 4 seconds. Find the time required by the particle in passing between points which are at distances 15 cm and 5 cm from the centre of force and are on the same side of it.

Solution:

Here $a = 20$ cm $\quad T = 4$ seconds

Since $T = \frac{2\pi}{\mu}$, $\mu = \frac{\pi}{2}$.

Let t_1 and t_2 seconds be the times when the particle is at distances 15 cm and 5 cm respectively from the centre of force.

Using $x = a\cos\mu t$, we have

$$15 = 20\cos\frac{\pi}{2}t_1 \Rightarrow t_1 = \frac{2}{\pi}\cos^{-1}\left(\frac{3}{4}\right)$$

and

$$5 = 20\cos\frac{\pi}{2}t_2 \Rightarrow t_2 = \frac{2}{\pi}\cos^{-1}\left(\frac{1}{4}\right).$$

$$\text{Required time} = t_2 - t_1 = \frac{2}{\pi}\left(\cos^{-1}\left(\frac{1}{4}\right) - \cos^{-1}\left(\frac{3}{4}\right)\right)$$
$$= 0.38 \text{ sec.}$$

2. A particle moving in a straight line with SHM has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2 respectively. Show that the period of motion is $2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

Solution:

The velocity v of the particle, when it is at a distance x from the mean position is given by $v^2 = \mu^2(a^2 - x^2)$, where a is the amplitude.

$$\therefore \; v_1^2 = \mu^2(a^2 - x_1^2) \rightarrow (1)$$

$$\text{and } v_2^2 = \mu^2(a^2 - x_2^2). \rightarrow (2)$$

Subtracting (1) from (2), we get

$$v_2^2 - v_1^2 = \mu^2(x_1^2 - x_2^2)$$

$$\Rightarrow \mu = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}.$$

Periodic time $T = \frac{2\pi}{\mu} = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

3. At the end of three successive seconds, the distances of a point moving with SHM from its mean position are x_1, x_2, x_3 respectively. Show that the time of a complete oscillation is $2\pi / \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$.

Solution:

Let the moving point be at distances x_1, x_2, x_3 from the mean position at the end of $t, t+1, t+2$ seconds respectively.

Using $x = a \cos \mu t$, we have

$$x_1 = a \cos \mu t \rightarrow (1)$$

$$x_2 = a \cos \mu(t+1) \rightarrow (2)$$

$$x_3 = a \cos \mu(t+2). \rightarrow (3)$$

Adding (1) and (3), we get $x_1 + x_3 = a\left[\cos\mu(t+2) + \cos\mu t\right]$

$$= a \cdot 2\cos\mu(t+1)\cos\mu$$

$$= 2x_2 \cos\mu \quad \left[\text{Using } (2)\right]$$

$$\Rightarrow \mu = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$

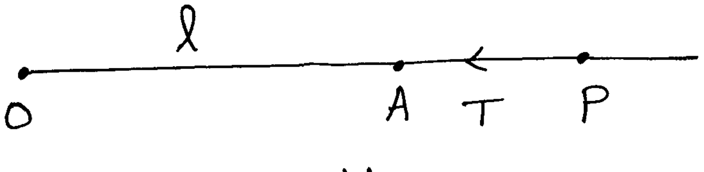
Hence the time of a complete oscillation is

$$\frac{2\pi}{\mu} = 2\pi \Big/ \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right).$$

4. In the case of a stretched elastic horizontal string which has one end fixed and a particle of mass m attached to the other, find the equation of motion of the particle given that l is the natural length of the string and e is its elongation due to a weight mg. Also find the displacements of the particle when initially $s = s_0$, $v = 0$.

Solution :

Let $OA = l$ be the elastic horizontal string with the end O fixed and a particle of mass m attached at A.



Let P be the position of the particle at any time t, when $OP = s$, so that the elongation $AP = s - l$.

Now, for the elongation e, tension $= mg$.

$\therefore$ For the elongation $(s-l)$, tension $= \dfrac{mg(s-l)}{e}$.

Since tension is the only horizontal force acting on the particle, its equation of motion is

$$m\frac{d^2s}{dt^2} = T$$

$$\Rightarrow m\frac{d^2s}{dt^2} = -\frac{mg(s-l)}{e}. \longrightarrow (1)$$

From (1), we have

$$\frac{d^2s}{dt^2} + \frac{g}{e}s = \frac{gl}{e}.$$

$\therefore$ The complete solution is,

$$s = A_1 \cos\left(\sqrt{\frac{g}{e}}\right)t + A_2 \sin\left(\sqrt{\frac{g}{e}}\right)t + l. \rightarrow (2)$$

(Verify!)

When $t=0$, $s = s_0$ so that from (2),

$$s_0 = A_1 + 0 + l$$

$$\Rightarrow A_1 = s_0 - l.$$

From (2), $\dfrac{ds}{dt} = -A_1\sqrt{\dfrac{g}{e}}\sin\left(\sqrt{\dfrac{g}{e}}\right)t + A_2\sqrt{\dfrac{g}{e}}\cos\left(\sqrt{\dfrac{g}{e}}\right)t.$

When $t=0$, $\dfrac{ds}{dt} = v = 0$. $\therefore A_2 = 0$.

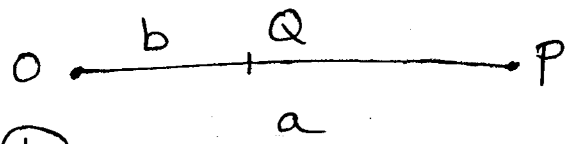
Therefore $s = (s_0 - l)\cos\left(\sqrt{\dfrac{g}{e}}\right)t + l.$

5. A point moves in a straight line towards a centre of force $\frac{\mu}{(\text{distance})^3}$, starting from rest at a distance 'a' from the centre of force. Show that the time of reaching a point distant 'b' from the centre of force is $\frac{a}{\sqrt{\mu}}\sqrt{a^2-b^2}$ and that its velocity then is $\frac{\sqrt{\mu}}{ab}\sqrt{a^2-b^2}$.

Solution :

O is the centre of force and the point starts from P, where $OP = a$.

We have to find out the time of reaching from P to Q and velocity at Q, where $OQ = b$.



Given $\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} \rightarrow (1)$

Multiplying (1) both sides by $2\frac{dx}{dt}$, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot 2\frac{dx}{dt}.$$

Integrating both sides, we get

$$\left(\frac{dx}{dt}\right)^2 = \int\left(\frac{-\mu}{x^3}\right)2\frac{dx}{dt}\,dt + c$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x^2} + c \longrightarrow (2)$$

Given that when $x=a$, velocity $\frac{dx}{dt}=0$.

$\therefore$ By (2), $c=-\mu a^{-2}$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\mu\left(x^{-2}-a^{-2}\right)=\mu\left(\frac{a^2-x^2}{a^2x^2}\right). \rightarrow (3)$$

$\therefore$ Velocity towards O at Q, i.e., at $x=b$ is

$$\left(\frac{dx}{dt}\right)_{x=b}=\frac{\sqrt{\mu}}{ab}\sqrt{a^2-b^2}, \left[\text{from } (3)\right].$$

Again, separting the variable in equation (3)

$$\frac{-ax}{\sqrt{a^2-x^2}}\,dx=\sqrt{\mu}\,dt$$

where negative sign is taken because point is moving towards O.

Integrating both sides, we get

$$\int\frac{-ax}{\sqrt{a^2-x^2}}\,dx=\int\sqrt{\mu}\,dt+A$$

$$\Rightarrow a\sqrt{a^2-x^2}=\sqrt{\mu}\,t+A.$$

When $x=a$, $t=0$. Hence $A=0$.

$$\therefore \text{At Q (when } x=b),\ t=\frac{a}{\sqrt{\mu}}\sqrt{a^2-b^2}.$$

6. In a system, the amplitude of motion is 5 metres and the period is 4 seconds. Find the time required by the particle in passing between the points which are at distances of 4 metres and 2 metres from the centre of force and are on same side of it. Also find the velocities of these points.

Solution :

Equation of SHM is $\frac{d^2x}{dt^2} = -\mu^2 x$. $\rightarrow$ (1)

Time period $T = \frac{2\pi}{\mu} = 4$, is given.

$$\Rightarrow \mu = \frac{\pi}{2}.$$

Since $a = 5$ m, the solution of (1) is

$$x = a\cos\mu t = 5\cos\frac{\pi}{2}t. \rightarrow (2)$$

Let t_1 sec. and t_2 sec. be the times when the particle is at a distance of 4 metres and 2 metres respectively from the centre of force. Then, from (2),

$$4 = 5\cos\frac{\pi}{2}t_1 \Rightarrow t_1 = \frac{2}{\pi}\cos^{-1}\left(\frac{4}{5}\right)$$

and $2 = 5\cos\frac{\pi}{2}t_2 \Rightarrow t_2 = \frac{2}{\pi}\cos^{-1}\left(\frac{2}{5}\right)$.

$\therefore$ Time required in passing through these points is, $t_2 - t_1 = \frac{2}{\pi}\left[\cos^{-1}\left(\frac{2}{5}\right) - \cos^{-1}\left(\frac{4}{5}\right)\right] = 0.33$ sec.

Differentiating (2) wrt 't', we get

$$\frac{dx}{dt} = -\frac{5\pi}{2}\sin\frac{\pi}{2}t = -\frac{5\pi}{2}\sqrt{1-\cos^2\frac{\pi t}{2}}$$

$$= -\frac{5\pi}{2}\sqrt{1-\frac{x^2}{25}} = -\frac{\pi}{2}\sqrt{25-x^2}.$$

When $x = 4$ m, $v = -\frac{\pi}{2}\sqrt{25-16} = -\frac{3\pi}{2}$ m/sec.

When $x = 2$ m, $v = -\frac{\pi}{2}\sqrt{25-4} = -\frac{\pi\sqrt{21}}{2}$ m/sec.

Negative sign indicates that it is directed towards centre of force.

7. A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P, where $OP = b$ with velocity v in the direction OP. Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$.

Solution :

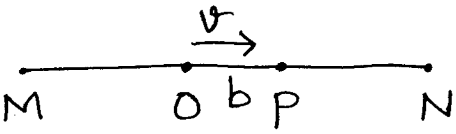
Let the amplitude be a.

Required time = time taken from P to N
+
time taken from N to P

$\Rightarrow$ Required time = 2 (time taken from N to P) $\rightarrow$ (1)

For the motion from N to P, we have

$$\frac{dx}{dt} = -\mu\sqrt{a^2 - x^2}$$



$$\Rightarrow dt = \frac{-dx}{\mu\sqrt{a^2-x^2}} \rightarrow (2)$$

Let t_1 be the time from N to P.

Then at N, $t = 0$, $x = a$ and at P, $t = t_1$, $x = b$.

$\therefore$ Integrating (2), we get

$$\int_0^{t_1} dt = \frac{1}{\mu}\int_a^b \frac{-dx}{\sqrt{a^2-x^2}}$$

$$t_1 = \frac{1}{\mu}\left[\cos^{-1}\left(\frac{x}{a}\right)\right]_a^b = \frac{1}{\mu}\cos^{-1}\left(\frac{b}{a}\right).$$

From (1), required time $= \dfrac{2}{\mu}\cos^{-1}\left(\dfrac{b}{a}\right)$

$$= \frac{2}{\mu}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)$$

Since $v = \dfrac{dx}{dt} = -\mu\sqrt{a^2-x^2}$ and at P, $x = b$,

$\dfrac{dx}{dt} = v$, we have $v = -\mu\sqrt{a^2-b^2} = \mu\sqrt{a^2-b^2}$. (neglect -ve sign)

$\therefore$ Required time $= \dfrac{2}{\mu}\tan^{-1}\left(\dfrac{v}{\mu b}\right)$.

$$= \frac{2}{\left(2\pi/T\right)}\tan^{-1}\left(\frac{v}{\frac{b\,2\pi}{T}}\right)$$

$$\left[\because \mu = \frac{2\pi}{T}\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

8. Show that if the displacement of a particle in a straight line is expressed by the equation $x = a\cos\mu t + b\sin\mu t$, it describes a simple harmonic motion whose amplitude is $\sqrt{a^2+b^2}$ and time period is $\dfrac{2\pi}{\mu}$.

Solution :

$$x = a\cos\mu t + b\sin\mu t \longrightarrow (1)$$

$$\therefore \frac{dx}{dt} = -a\mu\sin\mu t + b\mu\cos\mu t \longrightarrow (2)$$

$$\frac{d^2x}{dt^2} = -a\mu^2\cos\mu t - b\mu^2\sin\mu t = -\mu^2 x$$

~~which~~ That is, $\dfrac{d^2x}{dt^2} + \mu^2 x = 0$ which represents simple harmonic motion with centre at origin.

Time period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{\mu}$.

Also, amplitude is the value of x, when $\frac{dx}{dt}=0$.

From (2), $0 = -a\mu \sin \mu t + b\mu \cos \mu t$

$$\Rightarrow \tan \mu t = \frac{b}{a}$$

$$\therefore \sin \mu t = \frac{b}{\sqrt{b^2+a^2}} \text{ and } \cos \mu t = \frac{a}{\sqrt{b^2+a^2}}.$$

From (1), $x = a \cdot \frac{a}{\sqrt{a^2+b^2}} + b \cdot \frac{b}{\sqrt{a^2+b^2}}$

$$\Rightarrow x = \sqrt{a^2+b^2}.$$

This completes the proof.

9. A particle moves with SHM in a straight line under the action of a force which is proportional to the distance of the particle from $x=0$. If it starts at $x=5$ cm with a velocity of 10 cm/sec and it reaches an extreme position $x=10$ cm, at what speed does it pass through the origin?

Solution:

$$v^2 = \mu(a^2 - x^2) \rightarrow (1)$$

$$x = 5,\ v = 10.$$

$$\therefore 100 = \mu(a^2 - 25) \rightarrow (2)$$

Dividing (1) by (2), we get

$$\frac{v^2}{100} = \frac{a^2 - x^2}{a^2 - 25}$$

$$\Rightarrow \frac{v^2}{100} = \frac{100 - x^2}{100 - 25} \quad [\because a = 10]$$

$$\Rightarrow v^2 = \frac{4}{3}(100 - x^2)$$

At $x = 0$, $\quad v^2 = \frac{400}{3} = 133.33$

$$\therefore \; v = \sqrt{133.33} = 11.546 \text{cm/sec.}$$

Hence the required speed is 11.546 cm/sec.

10. A particle moves with SHM in a straight line. In the first second, starting from rest, it travels a distance 'a' and in the next second, it travels a distance 'b' in the same direction. Prove that the amplitude of motion is $\frac{2a^2}{3a-b}$.

Solution:

Let A be the amplitude of motion.

According to the problem, $A - a = A \cos \mu \rightarrow (1)$

$$\Rightarrow A - a - b = A \cos 2\mu \rightarrow (2)$$

Now, $A - a - b = A\left[2\cos^2 \mu - 1\right]$

$$= A\left[2\left(\frac{A-a}{A}\right)^2 - 1\right]$$

$$= \frac{2}{A}\left(A^2 + a^2 - 2Aa\right) - A$$

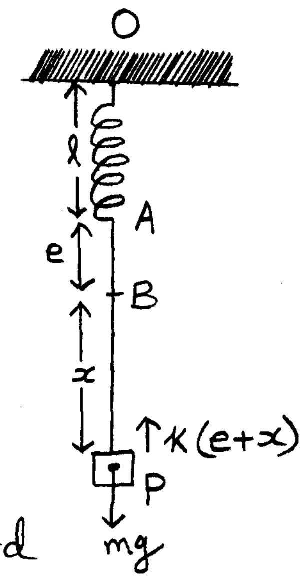
$$\Rightarrow A - a - b = 2A + \frac{2a^2}{A} - 4a - A$$

$$\Rightarrow 3a - b = \frac{2a^2}{A}$$

$$\Rightarrow A = \frac{2a^2}{3a-b}.$$

Other Oscillatory Motions

Free Oscillations



Let a spring OA of natural length l be suspended vertically from a fixed point O. Let a body of mass m be suspended from the end A and let e $(= AB)$ be the extension produced. The body is in equilibrium at B, under the action of its weight mg acting vertically downwards and the upward tension $T = ke$, where k is the stiffness (spring constant) of the spring, where $k = \frac{\lambda}{l}$, where λ is the modulus of elasticity of the spring. Thus $mg = ke$.

Let the body be displaced through a further distance from the equilibrium position B and then released. Clearly the body will begin to move upwards.

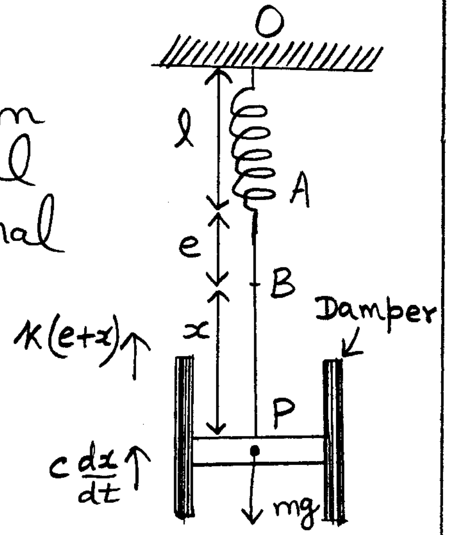
Let, at time t, the distance of the body be x $(= BP)$.

Then the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) = -kx \quad [\because mg = ke]$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x.$$

Damped Free Oscillations



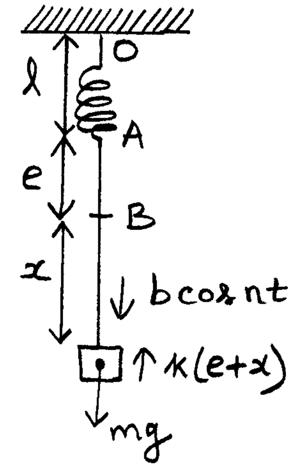
If the motion of the mass m is subjected to an additional force of resistance, proportional to the instantaneous velocity of the mass, say, $c\frac{dx}{dt}$, produced by a damper, the oscillations are said to be damped.

The equation of motion of the mass is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt}.$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{kx}{m} = 0.$$

Forced Oscillations without Damping



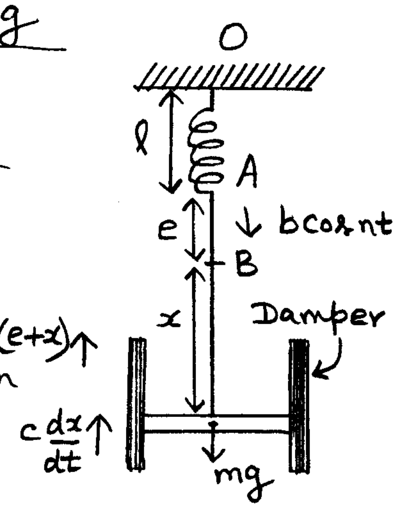
If the point of support of the spring is also vibrating with an external periodic force, then the resulting motion is called forced oscillatory motion.

If we assume that an external periodic force $b\cos nt$ is applied to the support B of the spring, the support is not steady and the equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) + b\cos nt$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{b}{m}\cos nt.$$

Forced Oscillations with Damping



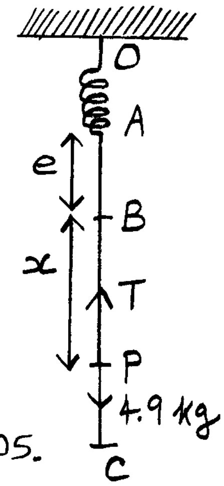
If we assume that an external periodic force $b \cos nt$ is applied to the support B of the spring and if there is a damping force, say, $c\frac{dx}{dt}$, then the equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt} + b\cos nt$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{b}{m}\cos nt.$$

11. A body weighing 4.9 kg is hung from a spring. A pull of 10 kg will stretch the spring to 5 cm. The body is pulled down 6 cm below the static equilibrium position and then released. Find the displacement of the body from its equilibrium position at time t seconds, the maximum velocity and the period of oscillation, where $g = 9.8\, m/sec^2$.

Solution :



Let O be the fixed end and A the free end of the spring. Let k be the stiffness of the spring (or the spring constant).

Since a pull of 10 kg stretches the spring by 0.05 metres, $10 = k \times 0.05$.

$\Rightarrow k = 200$ kg/m.

Let $e(=AB)$ be the elongation produced by the body weighing 4.9 kg hanging in equilibrium, then $4.9 = ke = 200e$. Hence $e = 0.0245$ m.

Now the body is pulled down to C, where $BC = 0.06$ m. After t sec. of its release from C, let the body be at P, where $BP = x$ metres. The forces acting on the body are

(i) its weight $W = 4.9$ kg downwards

(ii) the restoring force $T = k(e+x) = 200(0.0245 + x)$ kg upwards.

The equation of motion of the body is

$$m\frac{d^2x}{dt^2} = W - T$$

$$\Rightarrow \frac{4.9}{g}\frac{d^2x}{dt^2} = 4.9 - 200(0.0245 + x)$$

$$\Rightarrow \frac{4.9}{9.8}\frac{d^2x}{dt^2} = -200x$$

$$\Rightarrow \frac{d^2x}{dt^2} + 400x = 0$$

$$\therefore \; x = A\cos 20t + B\sin 20t.$$

When $t = 0$, $x = 0.06$ and $\frac{dx}{dt} = 0$.

$A = 0.06$ and $B = 0$.

Therefore $x = 0.06\cos 20t$ which gives the displacement of the body from its equilibrium position at time t.

Maximum velocity $= \mu \times$ amplitude

$= 20 \times 0.06 = 1.2$ m/sec.

Period of oscillation $= \frac{2\pi}{\mu} = \frac{\pi}{10} = 0.314$ sec.

12. A spring for which the spring constant $k = 700\ \mathrm{Nm^{-1}}$ hangs in a vertical position with its upper end fixed to a support. A mass of 20 kg is attached to the lower end and system brought to rest. Find the position of the mass at time t, if a force $70 \sin 2t$ N is applied to the support.

Solution :

The equation of motion is

$$m\frac{d^2x}{dt^2} = -kx + 70 \sin 2t$$

$$\Rightarrow 20\frac{d^2x}{dt^2} = -700x + 70 \sin 2t$$

$$\Rightarrow \frac{d^2x}{dt^2} + 35x = \frac{7}{2}\sin 2t.$$

Hence $x = A\cos\sqrt{35}\,t + B\sin\sqrt{35}\,t + \frac{7}{62}\sin 2t.$

At $t = 0$, $x = 0 = \frac{dx}{dt}$.

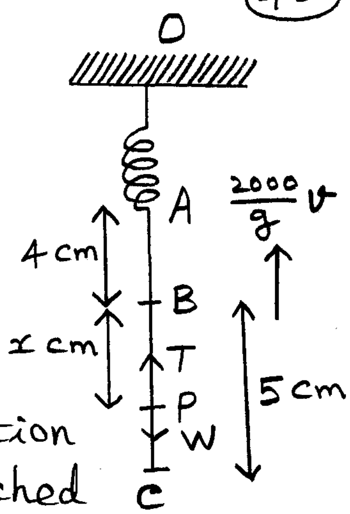
$A = 0$ and $B = \frac{-7}{31\sqrt{35}}$.

$$\therefore x = -\frac{7}{31\sqrt{35}}\sin\sqrt{35}\,t + \frac{7}{62}\sin 2t.$$

13. A mass of 200 gm is tied at the end of a spring which extends to 4 cm under a force 196,000 dynes. The spring is pulled 5 cm and released. Find the displacement, t seconds after release, if there be a damping force of 2000 dynes per cm per second.

Solution :

Let O be the fixed end and A the free end of the spring.



Since a force $\frac{196000}{980} = 200$ gm stretches the spring by 4 cm.

Hence $200 = k \times 4$.

$\Rightarrow k = 50$ gm/cm, where k is the restoring force.

Let B be the equilibrium position when a mass of 200 gm is attached to A, then $200 = k \times AB = 50 \times AB$

$\Rightarrow AB = 4$ cm.

Now the 200 gm weight is pulled down to C, where $BC = 5$ cm. If t seconds after its release from C, the weight be at P, where $BP = x$ cm, then the forces acting on it are (i) its weight $W = 200$ gm acting downwards

(ii) the tension $T = k \times AP = 50(4+x)$ gm upwards

(iii) the damping force $\frac{2000}{g}\frac{dx}{dt}$ gm upwards.

$\therefore$ The equation of motion is,

$$\frac{200}{g}\frac{d^2x}{dt^2} = W - T - \frac{2000}{g}\frac{dx}{dt} = 200 - 50(4+x) - \frac{2000}{g}\frac{dx}{dt}$$

$$\Rightarrow \frac{d^2x}{dt^2} + 10\frac{dx}{dt} + \frac{g}{4}x = 0.$$

Hence $x = e^{-5t}\left(A\cos\sqrt{220}\,t + B\sin\sqrt{220}\,t\right)$.

When $t=0$, $x = 0 = \frac{dx}{dt}$.

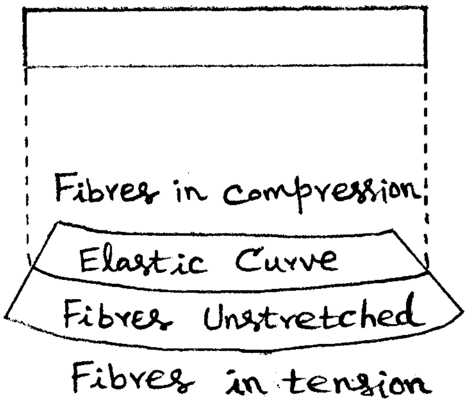
$A = 5$ and $B = \frac{25}{\sqrt{220}}$.

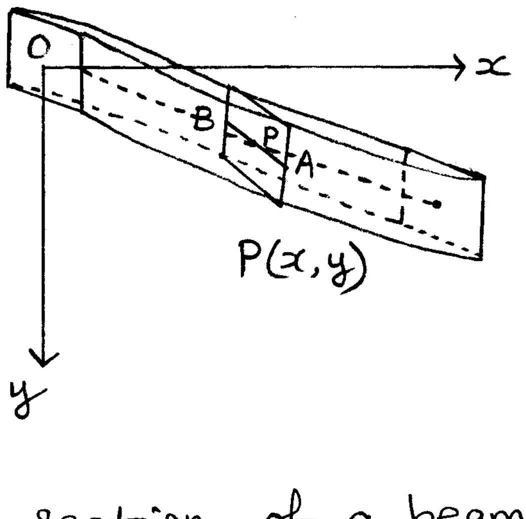
The displacement, t seconds after release, is given by

$$x = e^{-5t}\left(5\cos\sqrt{220}\,t + \frac{25}{\sqrt{220}}\sin\sqrt{220}\,t\right).$$

Bending of beams

Consider a beam which is made up of fibres running lengthwise. When the beam is bent it is clear that the fibres of the upper half are compressed and those of the lower half are stretched. Between these regions of compression and stretching, there is a layer which is neither compressed nor stretched. This surface is called the neutral surface of the beam and the curve of any particular fibre on this surface is known as an elastic curve or deflection curve of the beam. The line in which any plane section of beam cuts the neutral surface is called the axis of that section.





Consider a cross section of a beam at a distance x from the end. Let AB be the line of intersection with the neutral surface and P its intersection with the elastic curve.

Let $M =$ the moment of all external forces acting on either side of the two portions of the beam separated by the cross section about the line AB.

Let $E =$ modulus of elasticity of the beam,

$I =$ moment of inertia of the cross section about AB

and $R =$ radius of curvature of the elastic curve at $P(x, y)$.

Then the Bernoulli-Euler law states that

$$M = \frac{EI}{R}.$$

Now, $$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Since we assume that the deflection of the beam is small, $\left(\frac{dy}{dx}\right)^2$ can be neglected.

$$\therefore R = \frac{1}{d^2y/dx^2}$$

Thus $$M = EI \frac{d^2y}{dx^2} \longrightarrow (1)$$

Shear force (there is no bending moment) $= \frac{dM}{dx}$

$$= EI \frac{d^3y}{dx^3}$$

Intensity of loading $= \frac{d^2M}{dx^2}$

$$= EI \frac{d^4y}{dx^4}.$$

The bending moment M at the cross-section is the algebraic sum of the moments of the external forces acting on the segment of the beam about the line AB in the cross-section. We shall assume that upward forces give positive moments and downward forces give negative moments.

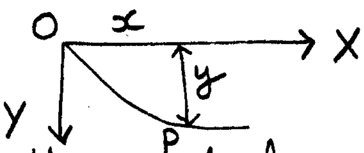
Boundary Conditions

The general solution of the differential equation (1) will contain two arbitrary constants. For a given problem, the arbitrary constants are to be determined from the following boundary (or end) conditions:

(i) End freely supported: At the freely supported end O, there is no deflection of the beam, so that $y=0$. Also there is no bending moment at this end, so that $\frac{d^2y}{dx^2}=0$.

Hence, at the freely supported end

$$y=0 \quad \text{and} \quad \frac{d^2y}{dx^2}=0.$$

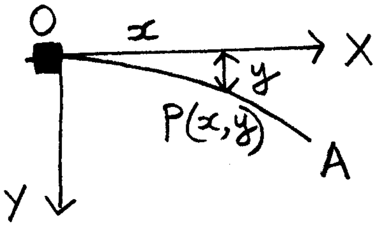


(ii) End fixed horizontal: At the end fixed horizontally, the deflection and the slope of the beam are zero.

$$\therefore y=0 \quad \text{and} \quad \frac{dy}{dx}=0.$$

(iii) End perfectly free: At the free end A in the following figure, there is no bending moment or shear force, so that

$$\frac{d^2y}{dx^2}=0, \text{ and } \frac{d^3y}{dx^3}=0.$$



Definitions

1. Beam or Bar: A rod of a circular or rectangular cross-section with its length very much greater than its thickness is called a beam.

2. Supported Beam: If a beam may just rest on support like a knife edge, then it is called a supported beam.

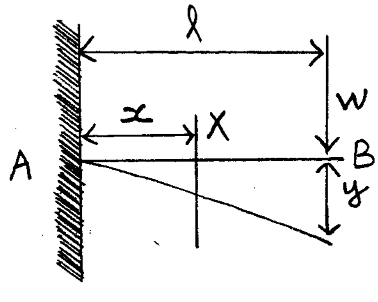
3. Fixed Beam: If one of the ends of a beam is firmly fixed, then it is called a fixed beam.

4. Cantilever: If the beam is fixed only at one end and loaded at the other, it is called a cantilever.

5. Strut: A beam of homogeneous isotropic material subject to compressive stress is called strut.

1\. A cantilever of length l is carrying a point load at the free end. Find the slope and deflection at the free end.

Solution:



The cantilever AB is shown in figure which is of length l and the end A is fixed and B is free. At the end B, a point of load 'w' is applied. Consider any section X of the cantilever a distance x from the fixed end A. The bending moment at this section is given by

$$EI\frac{d^2y}{dx^2} = -w(l-x) \longrightarrow (1)$$

[$\because$ moment $=$ force $\times$ distance

$= -w(l-x)$, since the force 'w' is acting downward we have $-$ve sign.]

Integrating (1) wrt x we get

$$EI\frac{dy}{dx} = -w\left(lx - \frac{x^2}{2}\right) + A. \longrightarrow (2)$$

The end A is fixed (clamped). At A, the slope $\frac{dy}{dx} = 0$ and $y = 0$ when $x = 0$.

When $x = 0$, $\frac{dy}{dx} = 0$.

$\therefore A = 0$

$$EI\frac{dy}{dx} = -w\left(lx - \frac{x^2}{2}\right). \longrightarrow (3)$$

Integrating (3) wrt x, we get

$$EI\, y = -w\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + B.$$

When $x=0$, $y=0$ (no deflection).

$$\therefore B = 0$$

Hence $$EIy = -w\left(\frac{lx^2}{2} - \frac{x^3}{6}\right). \longrightarrow (4)$$

Now we want the slope at the free end, that is, at B.

$$\frac{dy}{dx} = \frac{-w}{EI}\left(2\frac{lx}{2} - \frac{3x^2}{6}\right)$$

$$\left.\begin{array}{c}\text{slope at B}\\(x=l)\end{array}\right\} = \frac{-wl^2}{2EI}.$$

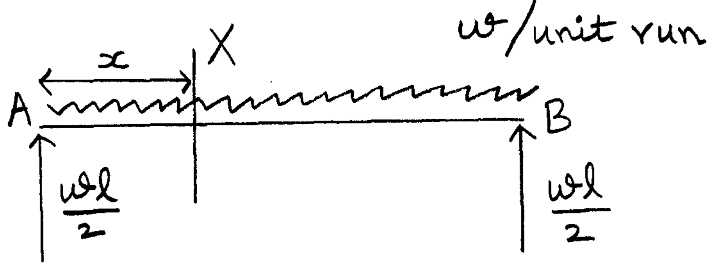
The deflection at B is obtained by putting $x=l$ in (4).

$$y = \frac{-wl^3}{3EI}.$$

2. A simply supported beam of span l carrying a uniformly distributed load of w per unit run over the whole span. Find the maximum deflection.

Solution :

The given beam is shown in figure. Vertical reaction at A and B is equal to $\frac{wl}{2}$ respectively.



Put $x = \frac{l}{2}$ in (4), we get

$$EI\,y = \frac{wl}{12}\frac{l^2}{4} - \frac{w}{24}\frac{l^4}{16} - \frac{wl^3}{24}\cdot\frac{l}{2}$$

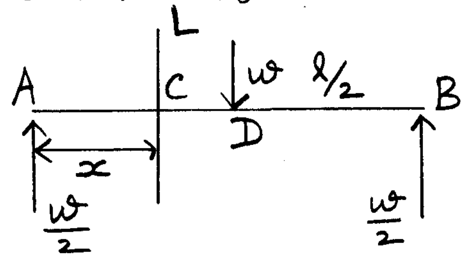
$$= \frac{5}{384} wl^4$$

$\therefore$ Maximum deflection $y = \frac{5}{384\,EI} wl^4$.

3. A horizontal beam of length l is freely supported at both ends and is loaded in the middle by a vertical load w. Show that the maximum deflection is $\frac{wl^3}{48\,EI}$.

Solution:

The beam given in the problem is shown in the figure.



Each vertical reaction $= \frac{w}{2}$.

The bending moment at any section L and C, distance x from the end A is given by

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}x \longrightarrow (1)$$

[In AC we have only one force at $A = \frac{w}{2}$. The moment of this force $\frac{w}{2}$ about x is given by $\frac{w}{2}\cdot x$. (moment = force × distance)]

Integrating (1) wrt x, we get

$$EI\frac{dy}{dx} = \frac{wx^2}{4} + A. \longrightarrow (2)$$

The maximum deflection occurs at the mid point where the load acts and at the point D the slope $\frac{dy}{dx} = 0$.

The bending moment at any section distant x from the end A is given by

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2}. \longrightarrow (1)$$

[The moment of force $\frac{wl}{2}$ about X is $\frac{wl}{2}x$.
The moment of force wx is $-wx.\frac{x}{2}$
which is the weight of the beam acting at a distance of $\frac{x}{2}$ from X.]

Integrating (1) wrt x, we get

$$EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} + A. \longrightarrow (2)$$

Since the loading is symmetrical, the maximum deflection will occur at mid span and hence the slope at mid span is zero. That is, the maximum slope occurs at $x=\frac{l}{2}$.

When $x=\frac{l}{2}$, $\frac{dy}{dx}=0$.

$$0 = \frac{wl}{4}.\frac{l^2}{4} - \frac{w}{6}.\frac{l^3}{8} + A$$

$$\Rightarrow A = -\frac{wl^3}{24}.$$

$$\therefore EI\frac{dy}{dx} = \frac{wlx^2}{4} - \frac{wx^3}{6} - \frac{wl^3}{24}. \rightarrow (3)$$

Integrating (3) wrt x, we get

$$EI.y = \frac{wlx^3}{12} - \frac{wx^4}{24} - \frac{wl^3}{24}x. \rightarrow (4)$$

The maximum deflection occurs at $x=\frac{l}{2}$.

When $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$.

$$0 = \frac{w}{4} \frac{l^2}{4} + A$$

$$\Rightarrow A = -\frac{wl^2}{16}.$$

Hence $EI \frac{dy}{dx} = \frac{w}{4} x^2 - \frac{wl^2}{16}$. $\rightarrow$ (3)

Integrating (3) wrt x, we get

$$EI\, y = \frac{wx^3}{12} - \frac{wl^2}{16} x + B$$

At A ($x=0$), the deflection is zero ($y=0$).

$$B = 0.$$

$$\therefore EI\, y = \frac{w}{12} x^3 - \frac{wl^2}{16} x.$$

The maximum deflection occurs at $x = \frac{l}{2}$.

$$\therefore y = \frac{1}{EI}\left[\frac{wl^3 - 3wl^3}{96}\right] = \frac{-wl^3}{48\, EI}$$

Maximum deflection $= \frac{-wl^3}{48EI}$.

4. A cantilever of length l carrying a uniformly distributed load w per unit run over the whole length. Find the slope equation and deflection equation of the cantilever. Find also the maximum deflection of the cantilever.

Solution:

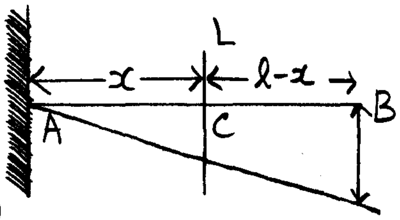
The cantilever is shown in the figure which is of length l and carrying uniformly distributed load 'w' per unit length.

The bending moment at any section L distance x from the end A is given by

$$EI\frac{d^2y}{dx^2} = -w\frac{(l-x)}{2}(l-x)$$

$$= -\frac{w}{2}(l-x)^2 \rightarrow (1)$$

[The load from A to C is $w(l-x)$ which is acting at the midway between A & C.



$\therefore$ The moment of this load (force) about A = − force × distance $= -w(l-x)\frac{(l-x)}{2}$, negative sign is for downward force.]

Integrating (1) wrt. x,

$$EI\frac{dy}{dx} = \frac{w}{6}(l-x)^3 + A$$

When $x = 0$ (fixed end), $\frac{dy}{dx} = 0$.

$$A = -\frac{wl^3}{6}.$$

$$EI\frac{dy}{dx} = \frac{w}{6}(l-x)^3 - \frac{wl^3}{6}. \rightarrow (2)$$

Integrating (2) wrt. x, we get

$$EI\,y = \frac{w}{6}\left(\frac{(l-x)^4}{-4}\right) - \frac{wl^3}{6}x + B.$$

When $x = 0$, $y = 0$

$$B = \frac{wl^4}{24}$$

$$\therefore\ EI\,y = -\frac{w}{24}(l-x)^4 - \frac{wl^3}{6}x + \frac{wl^4}{24}. \rightarrow (3)$$

The maximum deflection occurs at the free end A.
Put $x = l$ in (3), we get the maximum deflection.

$$EIy = -\frac{wl^4}{6} + \frac{wl^4}{24}$$

$$\Rightarrow y = -\frac{3wl^4}{8EI}.$$

5. A light horizontal strut AB is freely pinned at A and B. It is under the action of equal and opposite compressive force P at its ends and it carries a load w at its centre. Then for $0 < x < \frac{l}{2}$ $EI\frac{d^2y}{dx^2} + Py + \frac{1}{2}wx = 0$. Also $y = 0$ at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = \frac{l}{2}$. Prove that

$$y = \frac{w}{2P}\left(\frac{\sin nx}{n\cos\frac{nl}{2}} - x\right), \quad n^2 = \frac{P}{EI}.$$

<u>Solution:</u>

Given $EI\frac{d^2y}{dx^2} + Py = -\frac{wx}{2}$, is a second order differential equation.

The solution is,

$$y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x - \frac{wx}{2P}. \quad (\text{Verify!})$$

Given $x = 0$, $y = 0$.

$$A = 0$$

$$\frac{dy}{dx} = -A\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}\,x - \sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}\,x - \frac{w}{2P}$$

When $x = l$, $\frac{dy}{dx} = 0$.

$$B = \frac{w}{2P}\sqrt{\frac{EI}{P}}\ \sec\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}.$$

Therefore

$$y = \frac{w}{2P}\sqrt{\frac{EI}{P}}\ \sec\sqrt{\frac{P}{EI}}\cdot\frac{l}{2}\ \sin\sqrt{\frac{P}{EI}}\,x - \frac{wx}{2P}$$

$$= \frac{w}{2P}\left[\frac{\sin nx}{n\cos\left(\frac{nl}{2}\right)} - x\right], \text{ where } n^2 = \frac{P}{EI}.$$

6. A strut of length l is encastered at its lower end, its upper end is elastically supported against lateral deflection so that the resisting force is k times the end deflection and the deflection satisfies the following equation.

$$EI\frac{d^2y}{dx^2} = P(a-y) - ka(l-x).$$

Show that P is given by $\frac{\tan \alpha l}{\alpha l} = 1 - \frac{P}{kl}$, where $\alpha^2 = \frac{P}{EI}$.

Solution:

The given differential equation can be written as

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{Pa - ka(l-x)}{EI}. \longrightarrow (1)$$

The solution of (1) is

$$y = A\cos\alpha x + B\sin\alpha x + \frac{1}{P}\left\{Pa - ka(l-x)\right\},$$

(Verify!) $\left(\text{where } \alpha = \sqrt{\frac{P}{EI}}\right).$

$$\frac{dy}{dx} = -A\alpha \sin \alpha x + B\alpha \cos \alpha x + \frac{\kappa a}{P}.$$

At $x=0$, $y=0$ and $\frac{dy}{dx}=0$.

$$A = -\frac{a(P-\kappa l)}{P} \quad \text{and} \quad B = \frac{-\kappa a}{P\alpha}.$$

Therefore

$$y = \frac{-a(P-\kappa l)}{P}\cos \alpha x - \frac{\kappa a}{P\alpha}\sin \alpha x + \frac{1}{P}\left(Pa - \kappa a(l-x)\right).$$

At $x=l$, $y=a$.

Hence $$a = \frac{-a(P-\kappa l)}{P}\cos \alpha l - \frac{\kappa a}{P\alpha}\sin \alpha l + a$$

$$\Rightarrow \frac{\kappa a}{P\alpha}\sin \alpha l = \frac{-a(P-\kappa l)}{P}\cos \alpha l$$

$$\Rightarrow \tan \alpha l = \frac{-P\alpha}{\kappa} + \alpha l$$

$$\Rightarrow \frac{\tan \alpha l}{\alpha l} = 1 - \frac{P}{\kappa l}, \quad \alpha = \sqrt{\frac{P}{EI}}.$$

7. The differential equation satisfied by a beam uniformly loaded (W kg/m) with one end fixed and the other end subjected to tensile force P is given by $EI\frac{d^2y}{dx^2} = Py - \frac{1}{2}Wx^2$. Show that the elastic curve for the beam with conditions $y = 0 = \frac{dy}{dx}$ at $x=0$ is given by $y = \frac{W}{Pn^2}(1-\cosh nx) + \frac{Wx^2}{2P}$, where $n^2 = \frac{P}{EI}$.

Solution :

The given differential equation is

$$\frac{d^2y}{dx^2} - \frac{P}{EI} y = -\frac{W}{2EI} x^2. \rightarrow (1)$$

The solution of (1) is

$$y = Ae^{nx} + Be^{-nx} + \frac{W}{2EIn^2}\left(x^2 + \frac{2}{n^2}\right). \text{ (Verify!)}$$

$$\frac{dy}{dx} = nAe^{nx} - nBe^{-nx} + \frac{W}{2EIn^2}(2x)$$

When $x = 0$, $y = 0 = \frac{dy}{dx}$, $A = -\frac{W}{2EIn^4} = \frac{-W}{2n^2P} = B.$

$$\therefore y = \frac{-W}{2n^2P}\left(e^{nx} + e^{-nx}\right) + \frac{W}{2P}\left(x^2 + \frac{2}{n^2}\right)$$

$$\Rightarrow y = \frac{W}{Pn^2}(1 - \cosh nx) + \frac{W}{2P} x^2.$$

Electric Circuits

Basic elements of Electric Circuit:

The simplest electric circuit is a series circuit in which a source of electric energy (electromotive force) such as a generator or a battery and a resistor, which utilises energy, are components. If the switch is closed in the circuit, a current i will flow through the resistor and this will give a voltage drop, that is, the electric potential at the two ends of the resistor will be different. Using a voltameter, this voltage drop or the potential difference can be measured.

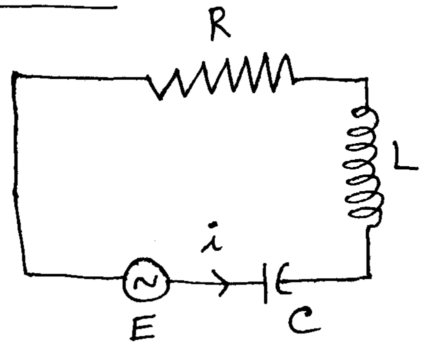
By ohm's law, the voltage drop E_R across a resistor is proportional to the instantaneous current i. That is, $E_R = Ri$, where R is the constant of proportionality called the Resistance of the resistor. The current i is measured in amperes. The resistance R is in ohms and the voltage E_R is in volts.

Two other important elements in a complicated circuits are capacitors and inductors. A capacitor is an element that stores energy. By experimental law, we have, the voltage drop E_C across a capacitor is proportional to the instantaneous electric charge q on the capacitor. That is, $E_C = \frac{q}{C}$, where C is called the capacitance of the capacitor and is measured in farads.

By the law of Faraday, the voltage drop E_L across an inductor is proportional to the instantaneous time rate of change of the current i. That is, $E_L = L\frac{di}{dt}$, where the constant of proportionality L is called the inductance of the inductor and is measured in henrys.

The time t is measured in seconds.

The algebraic sum of voltage (potential) drops across the elements of a closed circuit is equal to the applied voltage (electromotive force) in the circuit. This law is known as Kirchoff's law.



LCR Electrical Circuit

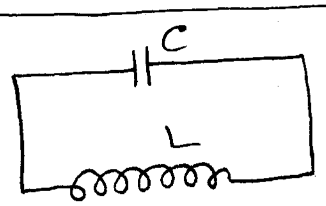
By the laws given above, the current i in an LCR electrical circuit composed of the resistance R, inductance L and capacitance C in series and acted on by an electromotive force E is given by

$$L\frac{di}{dt} + iR + \frac{q}{C} = E(t)$$

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t). \rightarrow (1)$$

Comparisons between electrical circuits and its equivalent mechanical systems

1. L–C Circuit with no emf | Free Oscillations



The eqn. (1) becomes

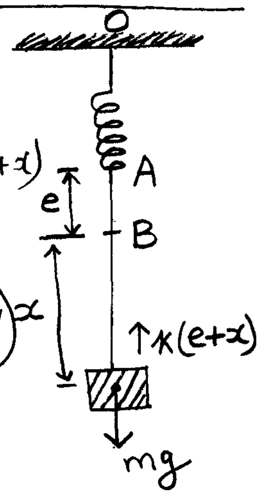
$$\frac{d^2q}{dt^2} = \frac{-1}{LC}q.$$

The eqn. is

$$m\frac{d^2x}{dt^2} = mg - k(e+x)$$

$$\Rightarrow \frac{d^2x}{dt^2} = \frac{-k}{m}x$$

(refer, page 37)

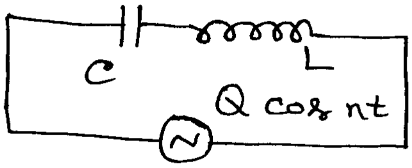


2. L-C Circuit with emf $Q \cos nt$

In L-C circuit with emf $Q\cos nt$, the eqn. (1) becomes

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = Q\cos nt$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = \frac{Q}{L}\cos nt.$$

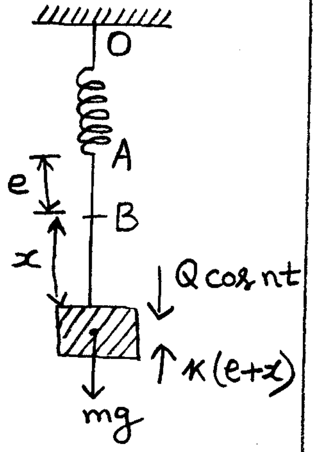


Forced Oscillations
Without Damping

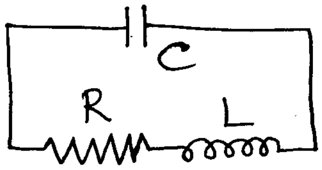
The equation of motion of mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) + Q\cos nt$$

$$\Rightarrow \frac{d^2x}{dt^2} = -kx + Q\cos nt$$



LCR Circuit without emf



The equation (1) becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

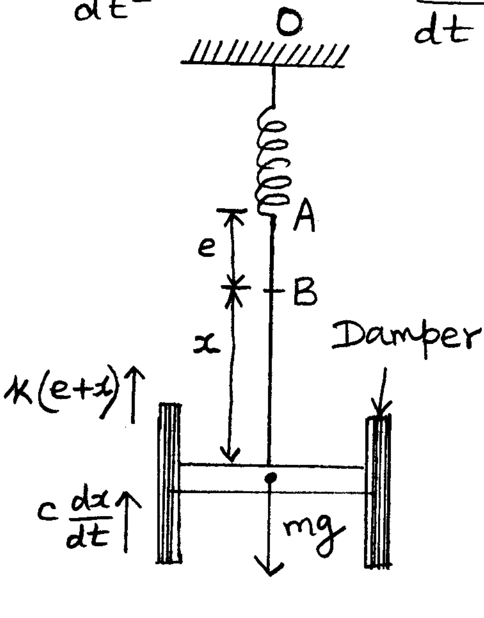
$$\Rightarrow \frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0.$$

Damped Free Oscillations

The equation of motion of the mass m is

$$m\frac{d^2x}{dt^2} = mg - k(e+x) - c\frac{dx}{dt}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}.$$

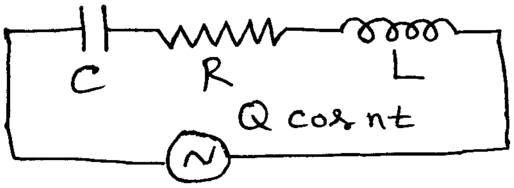


LCR Circuit with emf $Q \cos nt$

The equation (1) becomes

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=Q\cos nt$$

$$\Rightarrow \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{Q}{L}\cos nt.$$

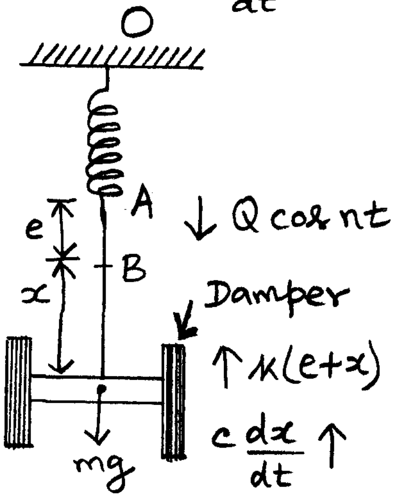


Forced Oscillations with damping

The equation of motion of mass m is

$$m\frac{d^2x}{dt^2}=mg-k(e+x)-c\frac{dx}{dt}+Q\cos nt$$

$$\Rightarrow \frac{d^2x}{dt^2}=-kx-c\frac{dx}{dt}+Q\cos nt.$$



1. The charge q on the plate of a capacitor is given by $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=E\sin\omega t$.
If the circuit is tuned so that $\omega^2=\frac{1}{LC}$, $R^2<\frac{4L}{C}$ and $q=\frac{dq}{dt}=0$ at $t=0$, then prove that $q=\frac{E}{R\omega}\left[e^{-\frac{Rt}{2L}}\left\{\cos pt+\frac{R}{2Lp}\sin pt\right\}-\cos\omega t\right]$ when $p^2=\frac{1}{LC}-\frac{R^2}{4L^2}$.

Solution:

The differential equation is,

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{E}{L}\sin\omega t. \rightarrow (1)$$

The solution of ① is,

$$q = e^{-\frac{Rt}{2L}}\left(A\cos pt + B\sin pt\right) - \frac{E}{R\omega}\cos\omega t.$$

(Verify!)

$$i = \frac{dq}{dt} = e^{-\frac{Rt}{2L}}\left(-Ap\sin pt + Bp\cos pt\right) - \frac{R}{2L}e^{-\frac{Rt}{2L}}\left(A\cos pt + B\sin pt\right) + \frac{E}{R}\sin\omega t.$$

At $t=0$, $q=0=i$.

$$A = \frac{E}{R\omega} \text{ and } B = \frac{E}{2Lp\omega}.$$

$$\therefore q = e^{-\frac{Rt}{2L}}\left[\frac{E}{R\omega}\cos pt + \frac{E}{2Lp\omega}\sin pt\right] - \frac{E}{R\omega}\cos\omega t$$

$$\Rightarrow q = \frac{E}{R\omega}\left[e^{-\frac{Rt}{2L}}\left\{\cos pt + \frac{R}{2Lp}\sin pt\right\} - \cos\omega t\right].$$

2. In a single closed electric circuit, the current i at time t is governed by the differential equation $E - Ri - L\frac{di}{dt} = 0$. Show that the current increases with time, and it approaches E/R as a limit, given that a constant emf E is impressed at time $t=0$, no current having flowed previously.

Solution: The given equation is

$$E - Ri = L\frac{di}{dt}$$

$$\Rightarrow \frac{dt}{L} = \frac{di}{E - Ri}$$

Initially, $i=0$ at $t=0$.

$$\Rightarrow \int_0^t \frac{dt}{L} = \int_0^i \frac{di}{E - Ri}$$

$$\Rightarrow \left[\log(E-Ri)\right]_0^i = -\frac{R}{L}\left[t\right]_0^t$$

$$\Rightarrow \log(E-Ri) - \log E = -\frac{R}{L}t$$

$$\Rightarrow \frac{E-Ri}{E} = e^{-\frac{R}{L}t}$$

$$\Rightarrow i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$

As t increases, $e^{-\frac{Rt}{L}}$ decreases and i also increases. The current approaches E/R when $t \to \infty$.

3. An emf $E \sin pt$ is applied at $t=0$ to a circuit containing a capacitance C and inductance L. The charge q is given by $L\frac{d^2q}{dt^2} + \frac{q}{C} = E \sin pt$. If $p^2 = \frac{1}{LC}$ and initially the current and charge are zero, find the current i at time t.

Solution:

The given differential equation is

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = E \sin pt. \quad \rightarrow (1)$$

The solution of (1) is,

$$q = A\cos pt + B\sin pt - \frac{E}{2Lp}t\cos pt.$$

$$i = \frac{dq}{dt} = -Ap\sin pt + Bp\cos pt - \frac{E}{2Lp}\left(\cos pt - pt\sin pt\right).$$

At $t=0$, $i=0=q$.

$$A = 0 \quad \text{and} \quad B = \frac{E}{2Lp^2}.$$

Therefore $i = \frac{E}{2L}t\sin pt.$

4. In the LCR circuit, the charge q is given by $\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{q}{LC} = 0$. If $t=0$, $q=Q$ and $i=0$ and $L = 2CR^2$, then prove that $q = Qe^{-kt}(\cos kt + \sin kt)$ if $2kRC = 1$.

Solution:

The differential equation is,

$$\frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{q}{LC} = 0.$$

The A.E. is $m^2 + \frac{1}{RC}m + \frac{1}{LC} = 0$

$$m = \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{LC}}}{2}$$

$$= \frac{-1}{2RC} \pm \sqrt{\frac{1}{4R^2C^2} - \frac{1}{LC}}$$

$$= -k \pm \sqrt{k^2 - \frac{1}{2C^2R^2}}$$

$$= -k \pm \sqrt{k^2 - 2k^2}$$

$$= -k \pm ik$$

$\therefore q = e^{-kt}(A\cos kt + B\sin kt)$

$i = \frac{dq}{dt} = e^{-kt}(-Ak\sin kt + Bk\cos kt) - ke^{-kt}(A\cos kt + B\sin kt)$

When $t=0$, $q=Q$ and $i=0$.

$A = Q = B.$

Hence $q = Qe^{-kt}(\cos kt + \sin kt).$

5. An inductor of 2 henries, resistor of 16 ohms, and capacitor of 0.02 farads are connected in series with a battery of emf $E = 100 \sin 3t$. At $t=0$, the charge on the capacitor and current in the circuit are zero. Find the charge and current at $t>0$.

Solution:

The differential equation of LCR circuit is given by

$$L \frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E$$

$$\Rightarrow 2\frac{d^2q}{dt^2} + 16\frac{dq}{dt} + \frac{q}{0.02} = 100 \sin 3t$$

$$\Rightarrow \frac{d^2q}{dt^2} + 8\frac{dq}{dt} + 25q = 50 \sin 3t. \quad \text{(1)}$$

The solution of (1) is,

$$q = e^{-4t}\left(A\cos 3t + B\sin 3t\right) + \frac{25}{52}\left(2\sin 3t - 3\cos 3t\right)$$

$$i = \frac{dq}{dt} = -4e^{-4t}\left(A\cos 3t + B\sin 3t\right) + e^{-4t}\left(3B\cos 3t - 3A\sin 3t\right) + \frac{25}{52}\left(9\sin 3t + 6\cos 3t\right).$$

When $t=0$, $i = 0 = q$.

$$A = \frac{75}{52} \text{ and } B = \frac{50}{52}.$$

$$\therefore\ q = \frac{25}{52}e^{-4t}\left(3\cos 3t + 2\sin 3t\right) + \frac{25}{52}\left(2\sin 3t - 3\cos 3t\right)$$

and $$i = \frac{75}{52}\left(2\cos 3t + 3\sin 3t\right) - \frac{25}{52}e^{-4t}\left(17\sin 3t + 6\cos 3t\right).$$